NEHRU COLLEGE OF ENGINEERING AND RESEARCH CENTRE
(NAAC Accredited)
(Approved by AICTE, Affiliated to APJ Abdul Kalam Technological University, Kerala)

DEPARTMENT OF ELECTRONICS AND COMMUNICATION ENGINEERING

COURSE MATERIALS

EC 303: Applied Electromagnetic Theory

VISION OF THE INSTITUTION

To mould true citizens who are millennium leaders and catalysts of change through excellence in education.

MISSION OF THE INSTITUTION

NCERC is committed to transform itself into a center of excellence in Learning and Research in Engineering and Frontier Technology and to impart quality education to mould technically competent citizens with moral integrity, social commitment and ethical values.

We intend to facilitate our students to assimilate the latest technological know-how and to imbibe discipline, culture and spiritually, and to mould them in to technological giants, dedicated research scientists and intellectual leaders of the country who can spread the beams of light and happiness among the poor and the underprivileged.

ABOUT DEPARTMENT

- Established in: 2002
- Course offered : B.Tech in Electronics and Communication Engineering

  M.Tech in VLSI
- Approved by AICTE New Delhi and Accredited by NAAC
- Affiliated to the A P J Abdul Kalam Technological University.

DEPARTMENT VISION

Provide well versed, communicative Electronics Engineers with skills in Communication systems with corporate and social relevance towards sustainable developments through quality education.

DEPARTMENT MISSION

1) Imparting Quality education by providing excellent teaching, learning environment.

2) Transforming and adopting students in this knowledgeable era, where the electronic gadgets (things) are getting obsolete in short span.

3) To initiate multi-disciplinary activities to students at earliest and apply in their respective fields of interest later.

4) Promoting leading edge Research & Development through collaboration with academia & industry.

PROGRAMME EDUCATIONAL OBJECTIVES

PEO1. To prepare students to excel in postgraduate programmes or to succeed in industry / technical profession through global, rigorous education and prepare the students to practice and innovate recent fields in the specified program/ industry environment.

PEO2. To provide students with a solid foundation in mathematical, Scientific and engineering fundamentals required to solve engineering problems and to have strong practical knowledge required to design and test the system.

PEO3. To train students with good scientific and engineering breadth so as to comprehend, analyze, design, and create novel products and solutions for the real life problems.

PEO4. To provide student with an academic environment aware of excellence, effective communication skills, leadership, multidisciplinary approach, written ethical codes and the life-long learning needed for a successful professional career.

PROGRAM OUTCOMES (POS)

Engineering Graduates will be able to:

1. **Engineering knowledge**: Apply the knowledge of mathematics, science, engineering fundamentals, and an engineering specialization to the solution of complex engineering problems.
2. **Problem analysis**: Identify, formulate, review research literature, and analyze complex engineering problems reaching substantiated conclusions using first principles of mathematics, natural sciences, and engineering sciences.
3. **Design/development of solutions**: Design solutions for complex engineering problems and design system components or processes that meet the specified needs with appropriate consideration for the public health and safety, and the cultural, societal, and environmental considerations.
4. **Conduct investigations of complex problems**: Use research-based knowledge and research methods including design of experiments, analysis and interpretation of data, and synthesis of the information to provide valid conclusions.
5. **Modern tool usage**: Create, select, and apply appropriate techniques, resources, and modern engineering and IT tools including prediction and modeling to complex engineering activities with an understanding of the limitations.
6. **The engineer and society**: Apply reasoning informed by the contextual knowledge to assess societal, health, safety, legal and cultural issues and the consequent responsibilities relevant to the professional engineering practice.
7. **Environment and sustainability**: Understand the impact of the professional engineering solutions in societal and environmental contexts, and demonstrate the knowledge of, and need for sustainable development.
8. **Ethics**: Apply ethical principles and commit to professional ethics and responsibilities and norms of the engineering practice.
9. **Individual and team work**: Function effectively as an individual, and as a member or leader in diverse teams, and in multidisciplinary settings.
10. **Communication**: Communicate effectively on complex engineering activities with the engineering community and with society at large, such as, being able to comprehend and write effective reports and design documentation, make effective presentations, and give and receive clear instructions.
11. **Project management and finance**: Demonstrate knowledge and understanding of the engineering and management principles and apply these to one’s own work, as a member and leader in a team, to manage projects and in multidisciplinary environments.
12. **Life-long learning**: Recognize the need for, and have the preparation and ability to engage in independent and life-long learning in the broadest context of technological change.

PROGRAM SPECIFIC OUTCOMES (PSO)

PSO1: Facility to apply the concepts of Electronics, Communications, Signal processing, VLSI, Control systems etc., in the design and implementation of engineering systems.
PSO2: Facility to solve complex Electronics and communication Engineering problems, using latest hardware and software tools, either independently or in team.optimization.

COURSE OUTCOMES
EC 303

SUBJECT CODE: EC 302	
COURSE OUTCOMES	
C303.1	Ability to apply basic mathematical concepts related to electromagnetic vector fields.
C303.2	Ability to apply Maxwell's equations in the analysis and application of electromagnetic fields.
C303.3	Ability to apply a solid foundation and a fresh perspective in the analysis and application of electromagnetic fields.
C303.4	Ability to analyze the propagation of electromagnetic waves in different media.
C303.5	Ability to analyze the characteristics of transmission lines and use Smith chart.
C303.6	Ability To apply the different modes of propagation in waveguides.

MAPPING OF COURSE OUTCOMES WITH PROGRAM OUTCOMES

CO'S	PO1	PO2	PO3	PO4	PO5	PO6	PO7	PO8	PO9	PO10	PO11	PO12
C303.1			3									1
C303.2	2	3	3		2			2				1
C303.3	2	3	3	3	2	3	2				2	1
C303.4		3	3	3		3						1
C303.5			3		2							1
C303.6			3	3			2					1
C303	2	3	3	3	2	3	2	2			2	1

CO'S	PSO1	PSO2	PSO3
C303.1			
C303.2	3		
C303.3	3	3	2
C303.4	3	3	2
C303.5			
C303.6			
C303	3	3	2

SYLLABUS

COURSE CODE	COURSE NAME	L-T-P-C	YEAR OF INTRODUCTION
EC 303	Applied Electromagnetic Theory	3-0-0-3	2015

Prerequisite: MA201 Linear Algebra & Complex Analysis, MA 101Calculus, MA 102 Differential equations

Course objectives:

The purpose of this course is:

1. To introduce basic mathematical concepts related to electromagnetic vector fields.
2. To impart knowledge on the basic concepts of electric and magnetic fields
3. To develop a solid foundation in the analysis and application of electromagnetic fields, Maxwell's equations and Poynting theorem.
4. To become familiar with propagation of signal through transmission lines and waveguides.

Syllabus:

Co-ordinate transformation, vector algebra, vector calculus, electrostatics, magneto statics, Maxwell's equations, Boundary condition, Solution of wave equation, propagation of plane EM wave in different media, Poynting vector theorem, transmission lines, Smith chart, Waveguides.

Expected outcome:

At the end of the course, students shall be able:

1. To develop a solid foundation and a fresh perspective in the analysis and application of electromagnetic fields.
2. To analyse the propagation of electromagnetic waves in different media.
3. To analyze the characteristics of transmission lines.
4. To understand the different modes of propagation in waveguides.

Text Books:

1. Mathew N O Sadiku, Elements of Electromagnetics, Oxford University Press, 6/e, 2014.
2. William, H., Jf Hayt, and John A. Buck. Engineering Electromagnetics. McGraw-Hill, 8/e McGraw-Hill, 2014.
3. John D. Kraus, Electromagnetics, 5/e, TMH, 2010.

References:

1. Joseph A Edminister , Electromagnetics, Schaum's Outline Series McGraw Hill, 4/e, 1995
2. Nannapaneni Narayana Rao, Elements of Engineering Electromagnetics, Pearson, 6/e, 2006.
3. Umran S. Inan and Aziz S. Inan, Engineering Electromagnetics, Pearson, 2010.
4. Martin A Plonus , Applied Electromagnetics, McGraw Hill, 2/e,1978.
5. Jordan and Balmain , Electromagnetic waves and Radiating Systems, PHI, 2/e,2013
6. Matthew N.O. Sadiku & S.V. Kulkarni "'Principles of Electromagnetics', Oxford University Press Inc. Sixth Edition, Asian Edition,2015

Course Plan			
Module	**Course content**	**Hours**	**Sem. Exam Marks**
I	Review of vector calculus, Spherical and Cylindrical coordinate system, Coordinate transformation	1	**0**
	Curl, Divergence, Gradient in spherical and cylindrical coordinate system.	1	
	Electric field – Application of Coulomb's law, Gauss law and Amperes current law (proof not required, simple problems only)	1	**15**
	Poisson and Laplace equations (proof not required, simple problems only), Determination of E and V using Laplace equation.	1	
	Derivation of capacitance and inductance of two wire transmission line and coaxial cable. Energy stored in Electric and Magnetic field.	2	
	Displacement current density, continuity equation. Magnetic vector potential. Relation between scalar potential and vector potential.	2	
II	Maxwell's equation from fundamental laws.	1	**15**
	Boundary condition of electric field and magnetic field from Maxwell's equations	1	
	Solution of wave equation	1	
	um, good conductor, media-attenuation, phase velocity, group velocity, skin depth.	3	
FIRST INTERNAL EXAM			
III	Reflection and refraction of plane electromagnetic waves at boundaries for normal & oblique incidence (parallel and perpendicular polarization), Snell's law of refraction, Brewster angle.	4	**15**
	Power density of EM wave, Poynting vector theorem, Complex Poynting vector.	3	
	Polarization of electromagnetic wave-linear, circular and elliptical polarisation.	2	
IV	Uniform lossless transmission line - line parameters	1	**15**
	Transmission line equations, Voltage and Current distribution of a line terminated with load	2	
	Reflection coefficient and VSWR. Derivation of input impedance of transmission line.	2	
SECOND INTERNAL EXAM			
V	Transmission line as circuit elements (L and C).	2	**20**
	Half wave and quarter wave transmission lines.	1	
	Development of Smith chart - calculation of line impedance and VSWR using smith chart.	2	
	Single stub matching (Smith chart and analytical method).	2	

VI	Parallel-Plate Waveguide - TE & TM waves.	1	**20**
	The hollow rectangular wave guide – modes of propagation of wave- dominant mode, group velocity and phase velocity -derivation and simple problems only.	3	
	Attenuation in wave guides, guide wavelength and impedance -derivation and simple problems only .	3	
END SEMESTER EXAM			

Question Paper

The question paper shall consist of three parts. Part A covers I and II module, Part B covers III and IV module, Part C covers V and VI module. Each part has three questions, which may have maximum four subdivisions. Among the three questions, one will be a compulsory question covering both modules and the remaining from each module, of which one to be answered. Mark patterns are as per the syllabus with 50 % for theory and 50% for logical/numerical problems, derivation and proof.

Question Paper Pattern (End Semester Exam) Maximum

Marks : 100 **Time : 3 hours**

The question paper shall consist of three parts. Part A covers modules I and II, Part B covers modules III and IV, and Part C covers modules V and VI. Each part has three questions uniformly covering the two modules and each question can have maximum four subdivisions. In each part, any two questions are to be answered. Mark patterns are as per the syllabus with 30% for theory and 70% for logical/numerical problems, derivation and proof.

QUESTION BANK

MODULE I				
Q:NO:	QUESTIONS	CO	KL	PAGE NO:
1	What is Electromagnetics? Provide its various applications.	CO1	K1	1
2	State and explain Gauss' Law using mathematical equations. Give its applications.	CO1	K2	18
3	Differentiate between Scalars, Vectors and Fields with the use of examples.	CO1	K2	2
4	Construct the Cylindrical Coordinate systems to mark a point P and explain the relevance of the variables.	CO1	K3	3
5	Define Curl of a Vector and provide its physical significance. State its properties.	CO1	K1	11
6	State and explain Coulomb's Law using mathematical equations. Give its applications.	CO1	K2	16
7	Define Gradient of a Scalar and Divergence of a Vector and provide its physical significances. State its properties.	CO1	K2	8
8	State and explain Stokes's Theorem with the aid of a diagram and mathematical equations.	CO1	K2	7
9	Define Laplacian of a Scalar and provide its physical significance. State its properties.	CO1	K2	14
10	State and explain Laplace's Law using mathematical equations. Determine **E** and V using this equation.	CO1	K3	28
11	Define Gradient of a Scalar and Curl of a Vector and provide its physical significances. State its properties.	CO1	K2	11
12	State and explain Divergence Theorem with the aid of a diagram and mathematical equations.	CO1	K3	27
13	State clearly the rules for transformation between Rectangular and Cylindrical Coordinate system and vice versa.	CO1	K2	3
14	State clearly the rules for transformation between Cartesian and Spherical Coordinate system and vice versa.	CO1	K2	3
15	State clearly the rules for transformation between Cylindrical and Spherical Coordinate system and vice versa.	CO1	K2	3
16	Differentiate between convection current and conduction current with the aid of examples.	CO1	K2	24

MODULE II				
1	Derive Maxwell's first equation from Faraday's law with the aid of a diagram.	CO2	K3	31
2	From first principles, derive Maxwell's second equation from Ampere's law. Represent it for time-harmonic signals.	CO2	K3	35
3	State and explain Maxwell's equations in both differential and integral forms. Represent it for time-harmonic signals.	CO2	K2	36
4	Describe Electrostatic Screening or Shielding	CO2	K2	44
5	Derive and explain Wave Equations using Maxwell's equations.	CO2	K3	48
6	Evaluate the Magnetic Boundary Conditions for a Dielectric-Dielectric interface with the help of diagrams.	CO2	K4	45
7	Evaluate the Electrostatic Boundary Conditions for a Conductor-Free Space interface with the help of a diagram.	CO2	K4	40
8	Evaluate the Electrostatic Boundary Conditions for a Dielectric-Conductor interface with the help of a diagram.	CO2	K4	41
9	Derive and explain Helmholtz Wave Equations using Maxwell's equations.	CO2	K3	48
10	Evaluate the Electric Boundary Conditions for a Dielectric-Dielectric interface with the help of diagrams.	CO2	K4	42
11	Evaluate the Electric Boundary Conditions for a Free Space-Conductor interface with the help of diagrams.	CO2	K4	43
12	Evaluate the Magnetostatic Boundary Conditions for a Dielectric-Dielectric interface with the help of a diagram.	CO2	K4	49
13	Define the following terms and give units: Phase velocity, Skin effect, Group velocity, Propagation Constant, Attenuation constant, Phase constant, Intrinsic Impedance	CO2	K1	54
14	Write down the complex relations for Phase constant and intrinsic impedance for a general medium. Construct the simplified equation for a lossless medium, free space and good conductor medium.	CO2	K3	55
MODULE III				
1	Analyze the Helmholtz's Wave Equations for EM Wave propagation in Lossy Dielectrics with the aid of Maxwell's equations for time harmonic signals and provide solution of the Wave Equations for both **E** field and **H** field.	CO3	K4	56
2	Analyze the propagation of plane waves in Lossless dielectrics.	CO3	K5	57
3	Analyze the propagation of plane waves in good conductors.	CO3	K4	58
4	State and explain Poynting's Theorem. From first principles, derive the equation governing this theorem for the power flow due	CO3	K3	59

	to EM waves.			
5	Define Skin Depth. Analyze the Skin resistances with the aid of diagram and equations.	CO3	K4	60
6	Investigate Reflection and Refraction of plane EM waves at boundaries for normal incidence, with the aid of diagrams and supporting mathematical analysis.	CO3	K3	64
7	Explore Reflection and Refraction of plane EM waves at boundaries for oblique incidence, with the aid of diagrams and supporting mathematical analysis.	CO3	K3	73
8	Using diagrams, describe the parallel and perpendicular polarization of plane waves.	CO3	K2	75
9	State and explain Snell's law of Refraction and Brewster angle.	CO3	K2	76
10	Analyze the linear polarization of EM waves, with the aid of diagrams.	CO3	K4	81
	MODULE IV			
1	State what is meant by a Transmission Line and what information is conveyed by it? What are the types, explain with neat diagrams, and clearly mention the area of application?	CO4	K2	89
2	Describe the Transmission Line parameters with the aid of diagrams, clearly mentioning the difference between lumped and distributed components. Obtain the mathematical relationship between them.	CO4	K2	90
3	From first principles, derive quantitatively the V and I Wave Equations (Transmission Line Equations), with the aid of diagrams. State the relevance of each and every term in the final expression.	CO4	K3	92
4	Analyse the solution of the V and I Wave Equations (Transmission Line Equations) to quantify Characteristic Impedence, Characteristic Admittance, and Propagation constant for a general lossy type Transmission line.	CO4	K2	94
5	Describe the two special cases of Transmission lines with the aid of supporting analysis to formulate the expressions for Characteristic Impedence, Propagation constant and phase velocity for each type.	CO4	K2	96
6	An air-line has characteristic impedance of 80 Ω and phase constant of 4 rad/m at 140 MHz. Calculate the inductance per meter and the capacitance per meter of the line.	CO4	K3	102
7	Explore Reflection and Refraction of plane EM waves at boundaries for oblique incidence, with the aid of diagrams and supporting mathematical analysis.	CO4	K3	103
8	Derive the general expression for the Input Impedence of a lossy transmission line in terms of its electrical length.	CO4	K3	104

9	Analyse the Reflection Coefficient and Standing Wave Ratio quantitatively for a transmission line.	CO4	K4	108
10	Derive the equation for Power on transmission line.	CO4	K3	110
11	Analyse the open circuited transmission line, short circuited transmission line and perfectly matched transmission lines.	CO4	K4	112
12	Analyse the half wave transmission line and quarter wave line and provide its application areas.	CO4	K4	114
	MODULE V			
1	Analyze the Smith Chart mathematically from first principles to obtain the closed loop equation for r-and z-circles, with the aid of succinct sketches. State the relevance of s-circles.	CO5	K4	119
2	A 30-m-long lossless transmission line with $Z_0 = 50\ \Omega$ operating at 2 MHz is terminated with a load $Z_L = 60 + j40\ \Omega$. If $u = 0.6c$ on the line, find (a) The reflection coefficient Γ (b) The standing wave ratio s (c) The input impedance	CO5	K3	144
3	Describe the Quarter wave Transformer matching used for Transmission lines with the aid of neat diagram.	CO5	K2	151
4	Analyse the Single Stub Tuner mathematically with the aid of diagram and provide qualitative details of applying the Smith chart with the aid of an appropriate sketch of the rough Smith chart.	CO5	K4	154
5	A 100 Ω lossless transmission line is to be matched to a load of 100-j80Ω, utilizing a shorted stub assuming an operating frequency of 20MHz and wave velocity of 0.6c, where c is the speed of light in vacuum.	CO5	K3	160
6	Use the Smith Chart to do this problem. Antenna with impedance $40 + j30\ \Omega$ is to be matched to a 100-Ω lossless line with a shorted stub. Determine (a) The required stub admittance (b) The distance between the stub and the antenna (c) The stub length (d) The standing wave ratio on each ratio of the system	CO5	K3	161
	MODULE VI			
1	Explain the Parallel-Plate Waveguide with the help of diagrams and give details of the EM wave propagation in it.	CO6	K2	169
2	Perform the detailed Analysis of Parallel-Plate Waveguide using Wave equations to get the expressions for **E** field and **H** field	CO6	K4	172

	strengths.			
3	A parallel-plate waveguide has plate separation *d* = 1 cm and is filled with Teflon having dielectric constant *r* = 2.1. Determine the maximum operating frequency such that only the TEM mode will propagate. Also find the range of frequencies over which the TE1 and TM1 (*m* = 1) modes, and no higher-order modes, will propagate.	CO6	K3	178
4	In the parallel-plate guide of Q3, the operating wavelength is λ = 2 mm. How many waveguide modes will propagate?	CO6	K3	178
5	In the guide of Q3, the operating frequency is 25 GHz. Consequently, modes for which m = 1 and m = 2 will be above cutoff. Determine the group delay difference between these two modes over a distance of 1 cm.	CO6	K3	179
6	Explain the qualitative details of hollow Rectangular Waveguide with the aid of simple diagram, how different is it from transmission line in principle and applications.	CO6	K2	181
7	Justify qualitatively and quantitatively the non-existence of TEM waves in hollow rectangular wave guides.	CO6	K2	184
9	Analyze the hollow Rectangular Waveguide to obtain the general expressions for the pertinent components of the **E** field and **H** Field strengths for Transverse Electric (TE) modes of propagation in the waveguide.	CO6	K4	185
10	Analyze the hollow Rectangular Waveguide to obtain the general expressions for the pertinent components of the **E** field and **H** Field strengths for Transverse Magnetic (TM) modes of propagation in the waveguide. Ponder upon the modes of propagation in the waveguide.	CO6	K4	192

APPENDIX 1

CONTENT BEYOND THE SYLLABUS

Applied Electromagnetic Theory

EC303 Module-1

Electromagnetics (EM)

- Electromagnetics (EM) is the study of the interactions between electric charges at rest and in motion.
- It involves the analysis, synthesis, physical interpretation, and application of electric and magnetic fields.
- Electromagnetics (EM) is a branch of Physics or Electrical Engineering in which electric and magnetic phenomena are studied.
- EM principles find applications in various related disciplines such as:
- microwaves, antennas, electric machines, satellite communications, bio-electromagnetics, plasmas, nuclear research, fiber optics, electromagnetic interference and compatibility, electromechanical energy conversion, radar meteorology, and remote sensing.

SCALARS AND VECTORS

- **A scalar** is a quantity that has only magnitude.
- Quantities such as time, mass, distance, temperature, entropy, electric potential, and population are scalars.
- A **vector** is a quantity that has both magnitude and direction.
- Vector quantities include velocity, force, displacement, and electric field intensity.
- represent a vector by a letter with an arrow on top of it, such as $\vec{A}$ and $\vec{B}$, or by a letter in boldface type such as **A** and **B**.
- A scalar is represented simply by a letter—e.g., A, B, U, and V.
- EM theory is essentially a study of some particular fields.

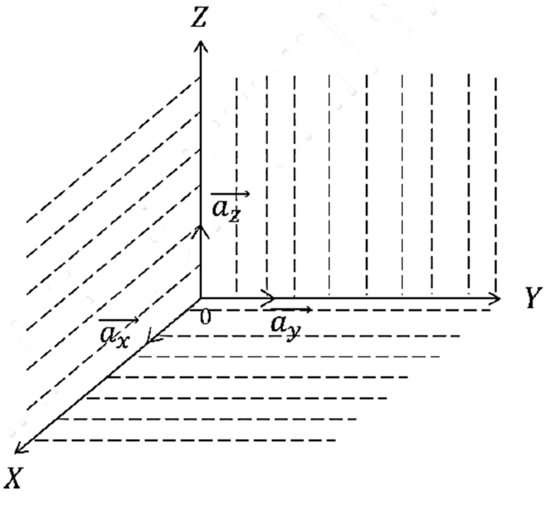
Field

- A **Field** is a function that specifies a particular quantity everywhere in a region.
- If the quantity is scalar (or vector), the field is said to be a scalar (or vector) field.
- Examples of scalar fields are temperature distribution in a building, sound intensity in a theater, electric potential in a region, and refractive index of a stratified medium.
- The gravitational force on a body in space and the velocity of raindrops in the atmosphere are examples of vector fields.

COORDINATE SYSTEMS & TRANSFORMATION

- An orthogonal system is one in which the coordinates are mutually perpendicular.
- The Cartesian, the Circular Cylindrical, and the Spherical are three different Coordinate systems.

Cartesian or Rectangular Co-ordinate system



- A point *P* can be represented as (*x, y, z*)
- The ranges of the coordinate variables *x, y,* and *z* are:

$$-\infty < x < \infty$$
$$-\infty < y < \infty$$
$$-\infty < z < \infty$$

- A vector A in Cartesian coordinates can be written as:

$$(A_x, A_y, A_z) \qquad \text{or} \qquad A_x\mathbf{a}_x + A_y\mathbf{a}_y + A_z\mathbf{a}_z$$

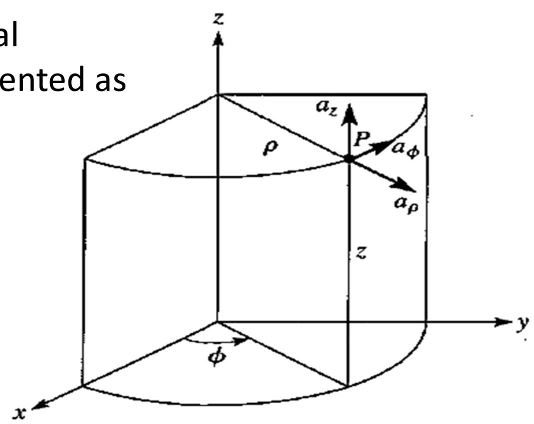
- where $\mathbf{a}_x$, $\mathbf{a}_y$, and $\mathbf{a}_z$ are unit vectors along the *x-, y-,* and *z*-directions.

CIRCULAR CYLINDRICAL COORDINATES (ρ, ϕ, z)

- This is applicable for with problems having cylindrical symmetry.

A point *P* in cylindrical coordinates is represented as

(ρ, ϕ, z)



- ρ is the <u>radius</u> of the cylinder passing through *P* or the radial distance from the z-axis.
- ϕ called the <u>azimuthal angle</u>, is measured from the x-axis in the xy-plane.
- and z is the same as in the Cartesian system.
- The ranges of the variables are:

$$0 \le \rho < \infty$$
$$0 \le \phi < 2\pi$$
$$-\infty < z < \infty$$

- A vector A in cylindrical coordinates can be written as:

$$(A_\rho, A_\phi, A_z) \qquad \text{or} \qquad A_\rho \mathbf{a}_\rho + A_\phi \mathbf{a}_\phi + A_z \mathbf{a}_z$$

- where $\mathbf{a}_\rho$, $\mathbf{a}_\phi$, and $\mathbf{a}_z$ are unit vectors in the ρ-, ϕ-, and z-directions.
- The relationships between the variables (x, y, z) of the Cartesian coordinate system and those of the cylindrical system (ρ, ϕ, z) :

$$\rho = \sqrt{x^2 + y^2}, \qquad \phi = \tan^{-1}\frac{y}{x}, \qquad z = z$$

$$x = \rho\cos\phi, \qquad y = \rho\sin\phi, \qquad z = z$$ Or

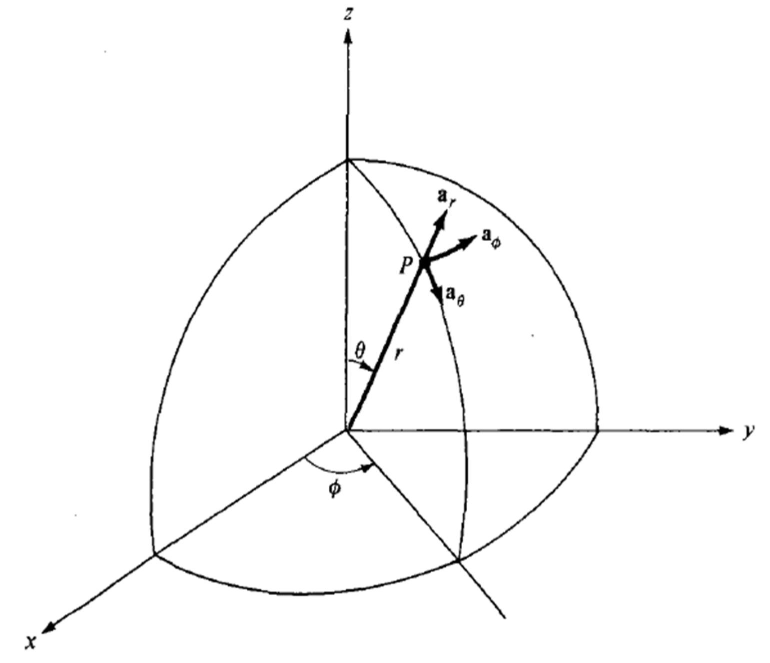
- In matrix form, the transformation of vector A from (A_x, A_y, A_z) to (A_ρ, A_ϕ, A_z) or vice versa as:

$$\begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} \qquad \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix}$$

SPHERICAL COORDINATES (r, θ, ϕ)

- Spherical coordinate system is used when dealing with problems with a degree of spherical symmetry.
- A point *P* can be represented as (r, θ, ϕ) and r is defined as distance from origin to point *P* or radius of a sphere centered at origin and passing thro' *P.*
- θ (called the <u>Colatitude</u>) is the angle between z-axis and the position vector of *P.*
- ϕ is measured from x-axis (the same <u>azimuthal angle</u> in cylindrical coordinates).
- The ranges of the variables are

$$0 \leq r < \infty$$
$$0 \leq \theta \leq \pi$$
$$0 \leq \phi < 2\pi$$



- A vector A in spherical coordinates may be written as:

$$(A_r, A_\theta, A_\phi) \qquad \text{or} \qquad A_r\mathbf{a}_r + A_\theta\mathbf{a}_\theta + A_\phi\mathbf{a}_\phi$$

- where $\mathbf{a}_r$, $\mathbf{a}_\theta$, and $\mathbf{a}_\phi$ are unit vectors along the r-, θ-, and ϕ- directions.
- The magnitude of **A** is $|\mathbf{A}| = (A_r^2 + A_\theta^2 + A_\theta^2)^{1/2}$
- The space variables *(*x, y, z*)* in Cartesian coordinates can be related to variables (r, θ, ϕ) of a spherical coordinate system:

$$r = \sqrt{x^2 + y^2 + z^2}, \qquad \theta = \tan^{-1}\frac{\sqrt{x^2 + y^2}}{z}, \qquad \phi = \tan^{-1}\frac{y}{x}$$

$$x = r\sin\theta\cos\phi, \qquad y = r\sin\theta\sin\phi, \qquad z = r\cos\theta$$

- In matrix form, the vector transformation is performed as follows:

$$\begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ -\cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix}$$

DEL(∇) OPERATOR

$$\nabla = \frac{\partial}{\partial x}\mathbf{a}_x + \frac{\partial}{\partial y}\mathbf{a}_y + \frac{\partial}{\partial z}\mathbf{a}_z$$ Cartesian coordinates

- This vector differential operator, or gradient operator, is not a vector but when it operates on a scalar function, a vector arises.
- The operator is useful to define:

1. The gradient of a scalar V, written as ∇V
2. The divergence of a vector $\mathbf{A}$, written as $\nabla \cdot \mathbf{A}$
3. The curl of a vector $\mathbf{A}$, written as $\nabla \times \mathbf{A}$
4. The Laplacian of a scalar V, written as $\nabla^2 V$

Gradient of a scalar

- The gradient of a scalar field V is a vector that represents both magnitude and direction of the max space rate of increase of V.

a. Cartesian co-ordinate

$$\nabla V = \frac{\partial V}{\partial x}\overrightarrow{a_x} + \frac{\partial V}{\partial y}\overrightarrow{a_y} + \frac{\partial V}{\partial z}\overrightarrow{a_z}$$

b. Cylindrical co-ordinate

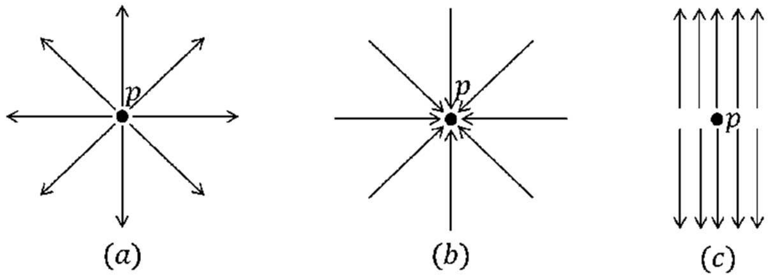
$$\nabla V = \frac{\partial V}{\partial \rho}\overrightarrow{a_\rho} + \frac{1}{\rho}\frac{\partial V}{\partial \phi}\overrightarrow{a_\phi} + \frac{\partial V}{\partial z}\overrightarrow{a_z}$$

c. Spherical co-ordinate

$$\nabla V = \frac{\partial V}{\partial r}\overrightarrow{a_r} + \frac{1}{r}\frac{\partial V}{\partial \theta}\overrightarrow{a_\theta} + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\overrightarrow{a_\phi}$$

Divergence

- Divergence of the vector field $\mathbf{A}$ at a given point is a measure of how much the field diverges or emanates from that point.



Positive divergence　　Negative divergence　　Zero divergence

- In case (a) the divergence of a vector field at point p is positive because the vector diverges at *p*.
- In case (b) the divergence of vector field at point *p* is negative because the vector converges at *p*.
- In case (c) the vector field has zero divergence at *p*.
- Divergence of a vector field represents the rate of change of the field strength in the direction of the field.

DIVERGENCE OF A VECTOR

∇

- The **divergence** of $\mathbf{A}$ at a given point P is the *outward* flux per unit volume as the volume shrinks about P.

$$\text{div } \mathbf{A} = \nabla \cdot \mathbf{A} = \lim_{\Delta v \to 0} \frac{\oint_S \mathbf{A} \cdot d\mathbf{S}}{\Delta v}$$

$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

Note the following properties of the divergence of a vector field:

1. It produces a scalar field (because scalar product is involved).
2. The divergence of a scalar V, div V, makes no sense.
3. $\nabla \cdot (\mathbf{A} + \mathbf{B}) = \nabla \cdot \mathbf{A} + \nabla \cdot \mathbf{B}$
4. $\nabla \cdot (V\mathbf{A}) = V\nabla \cdot \mathbf{A} + \mathbf{A} \cdot \nabla V$

The divergence theorem states that the total outward flux of a vector field $\mathbf{A}$ through the closed surface S is the same as the volume integral of the divergence of $\mathbf{A}$.

$$\oint_S \mathbf{A} \cdot d\mathbf{S} = \int_v \nabla \cdot \mathbf{A} \, dv$$

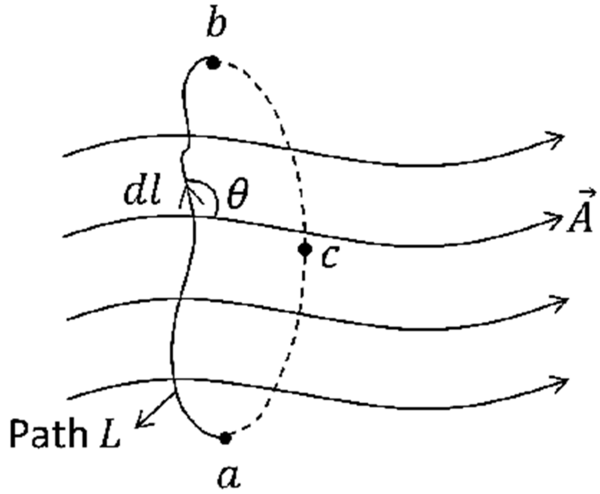
CURL OF A VECTOR

The curl of $\mathbf{A}$ is an axial or rotational vector whose magnitude is max circulation of $\mathbf{A}$ per unit area as area tends to zero and whose direction is normal direction of area when the area is oriented so as to make circulation max.

$$\text{curl } \mathbf{A} = \nabla \times \mathbf{A} = \left(\lim_{\Delta S \to 0} \frac{\oint_L \mathbf{A} \cdot d\mathbf{l}}{\Delta S} \right)_{\max} \mathbf{a}_n$$

Where the area ΔS is bounded by the curve L and $\mathbf{a}_n$ is the unit vector normal to the surface ΔS.

$$\nabla \times \mathbf{A} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$



- The curl provides the max value of the circulation of the field per unit area and indicates the direction along which this max value occurs.
- The curl of a vector field $\mathbf{A}$ at a point P may be regarded as a measure of the circulation or how much the field curls around P .

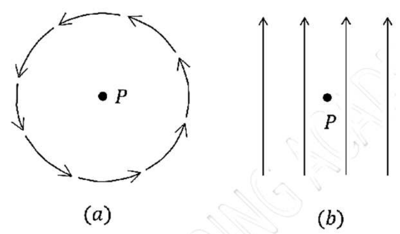


Illustration of curl

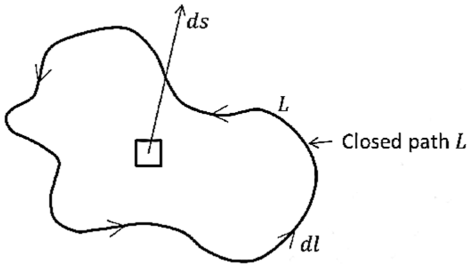
a) Curl at P points out of the page.
b) Curl at P is zero.

Note the following properties of the curl:

1. The curl of a vector field is another vector field.
2. The curl of a scalar field V, $\nabla \times V$, makes no sense.
3. $\nabla \times (\mathbf{A} + \mathbf{B}) = \nabla \times \mathbf{A} + \nabla \times \mathbf{B}$
4. $\nabla \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A}) + (\mathbf{B} \cdot \nabla\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B}$
5. $\nabla \times (V\mathbf{A}) = V\nabla \times \mathbf{A} + \nabla V \times \mathbf{A}$
6. The divergence of the curl of a vector field vanishes, that is, $\nabla \cdot (\nabla \times \mathbf{A}) = 0$.
7. The curl of the gradient of a scalar field vanishes, that is, $\nabla \times \nabla V = 0$.

Stoke's theorem

- Stoke's theorem states that the circulation of a vector field $\mathbf{A}$ around a (closed) path *L* is equal to the surface integral of the curl of $\mathbf{A}$ over the open surface *S* bounded by *L* provided that $\mathbf{A}$ and $\nabla\times\mathbf{A}$ are continuous on *S*.



$$\oint_L \mathbf{A}\cdot d\mathbf{l} = \int_S (\nabla\times\mathbf{A})\cdot d\mathbf{S}$$

- The direction of d**l** and d**S** must be chosen using the right-hand rule or right-handed screw rule.
- Using the right-hand rule, if we let the fingers point in the direction *of* d**l** *,* the thumb will indicate the direction of d**S** *.*
- Note that whereas the divergence theorem relates a surface integral to a volume integral, Stokes's theorem relates a line integral (circulation) to a surface integral.

Laplacian of a Scalar

- The Laplacian of a scalar field *V* written as $\nabla^2 V$
- It is the divergence of the gradient of V.

Thus, in Cartesian coordinates,

Laplacian $V = \nabla \cdot \nabla V = \nabla^2 V$

$$= \left[\frac{\partial}{\partial x}\mathbf{a}_x + \frac{\partial}{\partial y}\mathbf{a}_y + \frac{\partial}{\partial z}\mathbf{a}_z\right] \cdot \left[\frac{\partial V}{\partial x}\mathbf{a}_x + \frac{\partial V}{\partial y}\mathbf{a}_y + \frac{\partial V}{\partial z}\mathbf{a}_z\right]$$

$$\therefore \quad \nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

- Notice that the Laplacian of a scalar field is another scalar field.
- A scalar field *V* is said to be Harmonic in a given region if its Laplacian vanishes in that region. i.e., $\nabla^2 V = 0$
- This is called Laplace's equation.
- Laplacian of a vector **A** is given by:

$$\nabla^2 \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla \times \nabla \times \mathbf{A}$$

- This is defined as the gradient of the divergence of **A** minus the curl of the curl of **A**.
- In the Cartesian system→

$$\nabla^2 \mathbf{A} = \nabla^2 A_x \mathbf{a}_x + \nabla^2 A_y \mathbf{a}_y + \nabla^2 A_z \mathbf{a}_z$$

Simple Problems….

Find the Laplacian of the scalar field

$$V = e^{-z} \sin 2x \cosh y$$

Sol:

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

$$= \frac{\partial}{\partial x}(2e^{-z} \cos 2x \cosh y) + \frac{\partial}{\partial y}(e^{-z} \sin 2x \sinh y)$$

$$+ \frac{\partial}{\partial z}(-e^{-z} \sin 2x \cosh y)$$

$$= -4e^{-z} \sin 2x \cosh y + e^{-z} \sin 2x \cosh y + e^{-z} \sin 2x \cosh y$$

$$= -2e^{-z} \sin 2x \cosh y$$

Exercise…

Determine the Laplacian of the scalar fields

i. $V_1 = x^3 + y^3 + z^3$

ii. $U = x^2 y + xyz$

Coulomb's Law

- **Coulomb's law** states that the force between two point charges Q_1, and Q_2 is:
- 1. Along the line joining them
- 2. Directly proportional to the product $Q_1.Q_2$ of the charges.
- 3. Inversely proportional to the square of the distance *R* between them.

k→ proportionality constant.

$$F = \frac{k\,Q_1Q_2}{R^2}$$

$$k = \frac{1}{4\pi\varepsilon_o}$$

permittivity of free space (in farads per meter)

$$\varepsilon_o = 8.854 \times 10^{-12} \simeq \frac{10^{-9}}{36\pi}\ \text{F/m}$$

$$\text{or } k = \frac{1}{4\pi\varepsilon_o} \simeq 9 \times 10^9\ \text{m/F}$$

$$F = \frac{Q_1Q_2}{4\pi\varepsilon_o R^2}$$

$$\mathbf{F}_{12} = \frac{Q_1Q_2\,(\mathbf{r}_2 - \mathbf{r}_1)}{4\pi\varepsilon_o|\mathbf{r}_2 - \mathbf{r}_1|^3}$$

Electric field strength

- The electric field intensity (or electric field strength) E is the force per unit charge when placed in the electric field. $\mathbf{E} = \frac{\mathbf{F}}{Q}$
- The electric field intensity E is obviously in the direction of the force F and is measured in N/Coul or Volts/meter.
- The electric field intensity at point r due to a point charge located at r‘:

$$\mathbf{E} = \frac{Q}{4\pi\varepsilon_o R^2}\,\mathbf{a}_R = \frac{Q(\mathbf{r} - \mathbf{r}')}{4\pi\varepsilon_o|\mathbf{r} - \mathbf{r}'|^3}$$

ELECTRIC FLUX DENSITY

- The electric flux density is dependent on the medium in which the charge is placed and is defined by: $\mathbf{D} = \varepsilon_o\mathbf{E}$
- Define *electric flux* f in terms of **D**:

$$\Psi = \int \mathbf{D} \cdot d\mathbf{S}$$

- The electric flux is measured in coulombs, the vector field **D** is called the *electric flux density* and is measured in coulombs per square meter.

Gauss‘s Law

- Gauss's **law** stales that the total electric flux Ψ through any *closed* surface is equal to the total charge enclosed by that surface.

$$\Psi = Q_{\text{enc}}$$

$$\Psi = \oint d\Psi = \oint_S \mathbf{D} \cdot d\mathbf{S}$$

$$= \text{Total charge enclosed } Q = \int \rho_v \, dv$$

$$Q = \oint_S \mathbf{D} \cdot d\mathbf{S} = \int_v \rho_v \, dv$$

By applying divergence theorem:

$$\therefore \quad \oint_S \mathbf{D} \cdot d\mathbf{S} = \int_v \nabla \cdot \mathbf{D} \, dv$$

$$\therefore \quad \rho_v = \nabla \cdot \mathbf{D}$$

The volume charge density is the same as the divergence of the electric flux density.

Electric Potential

- The electric field intensity **E** due to a charge distribution can be obtained from Coulomb's law in general or from Gauss's law when the charge distribution is symmetric.
- Another way of obtaining **E** is from the electric scalar potential *V*
- This way of finding **E** is easier because it is easier to handle scalars than vectors.

- To move a point charge *Q* from one point to another in an electric field **E**, from Coulomb's law, the force on *Q* is F = *Q***E** so that the work done in displacing the charge by d**l** is:
- dW = - **F**. d**l**= -Q**E**.d**l**
- The negative sign indicates that the work is being done by an external agent.
- Thus the total work done, or the potential energy required, in moving *Q* from A to *B* is:

$$W = -Q \int_A^B \mathbf{E} \cdot d\mathbf{l} \qquad V_{AB} = \frac{W}{Q} = -\int_A^B \mathbf{E} \cdot d\mathbf{l}$$

- if the **E** field is due to a point charge *Q* located at the origin, then:

$$\mathbf{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\,\mathbf{a}_r$$

$$\therefore \quad V_{AB} = -\int_{r_A}^{r_B} \frac{Q}{4\pi\varepsilon_0 r^2}\,\mathbf{a}_r \cdot dr\,\mathbf{a}_r$$

$$= \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{r_B} - \frac{1}{r_A}\right]$$

$$\therefore \quad V_{AB} = V_B - V_A$$

- Thus if $V_A = 0$ as $r_A \rightarrow \infty$ the potential at any point (r_B —> r) due to a point charge *Q* located at the origin is:

$$V = \frac{Q}{4\pi\varepsilon_0 r}$$

- the potential at a distance r from the point charge is the work done per unit charge by an external agent in transferring a test charge from infinity to that point.

$$\therefore \quad V = -\int_{\infty}^{r} \mathbf{E} \cdot d\mathbf{l}$$

- Potential difference between points *A* and *B* is independent of the path taken.

$$\therefore \quad V_{BA} = -V_{AB}$$

$$\therefore \quad V_{BA} + V_{AB} = \oint \mathbf{E} \cdot d\mathbf{l} = 0$$

$$\therefore \quad \oint \mathbf{E} \cdot d\mathbf{l} = 0$$

- The line integral of **E** along a closed path must be zero.
- Physically, this implies that no net work is done in moving a charge along a closed path in an electrostatic field.

- Applying Stokes's theorem:

$$\therefore \quad \oint \mathbf{E} \cdot d\mathbf{l} = \int (\nabla \times \mathbf{E}) \cdot d\mathbf{S} = 0 \quad \Rightarrow \quad \nabla \times \mathbf{E} = 0$$

- Any vector field that satisfies this relation is said to be conservative, or irrotational.
- Thus an electrostatic field is a conservative field. Equation is referred to as *Maxwell's equation* for static electric fields.

$$dV = -\mathbf{E} \cdot d\mathbf{l} = -E_x\,dx - E_y\,dy - E_z\,dz$$

$$\therefore \quad dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz$$

Comparing the two expressions for *dV,* we obtain

$$E_x = -\frac{\partial V}{\partial x}, \qquad E_y = -\frac{\partial V}{\partial y}, \qquad E_z = -\frac{\partial V}{\partial z}$$

$$\therefore \ \mathbf{E} = -\nabla V$$

The electric field intensity is the gradient of *V.*
The negative sign direction of **E** is opposite to the direction in which *V* increases.
E is directed from higher to lower levels of *V.*

Poisson‘s And Laplace‘s Equations

- Poisson's and Laplace's equations are derived from Gauss's law.

$$\nabla \cdot \mathbf{D} = \nabla \cdot \varepsilon\mathbf{E} = \rho_v$$

$$\mathbf{E} = -\nabla V$$

$$\therefore \quad \nabla \cdot (-\varepsilon \nabla V) = \rho_v$$

$$\therefore \quad \nabla^2 V = -\frac{\rho_v}{\varepsilon}$$

This is known as **Poisson's equation**.

A special case of this equation occurs when $\rho_v = 0$ (i.e., for a charge-free region)

$$\therefore \quad \nabla^2 V = 0$$

This is known as **Laplace's equation**

$$\therefore \quad \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

- Laplace's equation is important in solving electrostatic problems involving a set of conductors maintained at different potentials.
- Examples include capacitors and vacuum tube diodes.
- Laplace's and Poisson's equations are also useful in various other field problems.

- For example, *V* would be interpreted as:
    - magnetic potential in Magnetostatic
    - temperature in heat conduction,
    - stress function in fluid flow,
    - pressure head in seepage.

Convection and Conduction Currents

- The current (in amperes) through a given area is the electric charge passing through the area per unit time.

$$I = \frac{dQ}{dt}$$

- Thus in a current of one ampere, charge is being transferred at a rate of one coulomb/sec.
- We now introduce the concept of *current density* **J**.
- If current ΔI flows through a surface ΔS*,* the current density is:

$$J_n = \frac{\Delta I}{\Delta S}$$

$$\Delta I = J_n \Delta S$$

assuming that the current density is perpendicular to the surface.
If the current density is not normal to the surface,

$$\Delta I = \mathbf{J} \cdot \Delta S$$

Thus, the total current flowing through a surface **S** is

$$I = \int_S \mathbf{J} \cdot d\mathbf{S}$$

Convection current, as distinct from conduction current, does not involve conductors and consequently does not satisfy Ohm's law.
It occurs when current flows through an insulating medium such as liquid, rarefied gas, or a vacuum.
A beam of electrons in a vacuum tube, for example, is a convection current.

The current I is the convection current and **J** is the convection current density in amperes/square meter (A/m^2).

- Conduction current requires a conductor.
- A conductor is has large amount of free electrons that provide conduction current due an impressed electric field.

Thus the conduction current density is:

$$\mathbf{J}=\sigma\mathbf{E}$$

where σ is the conductivity of the conductor.
The relationship is known as the point form of **Ohm's law**.

CONTINUITY EQUATION

Due to the principle of charge conservation, the time rate of decrease of charge within a given volume must be equal to the net outward current flow through the closed surface of the volume.

Thus current I_{out} coming out of the closed surface is

$$I_{\text{out}} = \oint \mathbf{J} \cdot d\mathbf{S} = \frac{-dQ_{\text{in}}}{dt} \qquad \text{..................(1)}$$

where Q_{in} is the total charge enclosed by the closed surface.

Using Divergence theorem,

$$\oint_S \mathbf{J} \cdot d\mathbf{S} = \int_v \nabla \cdot \mathbf{J}\, dv \qquad \text{………………..(2)}$$

$$\frac{-dQ_{\text{in}}}{dt} = -\frac{d}{dt}\int_v \rho_v\, dv = -\int_v \frac{\partial \rho_v}{\partial t}\, dv \qquad \text{………………..(3)}$$

Substitute eqns (2) and (3) in eqn (1) gives:

$$\int_v \nabla \cdot \mathbf{J}\, dv = -\int_v \frac{\partial \rho_v}{\partial t}\, dv$$

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho_v}{\partial t}$$ **Continuity equation**.

Terms and Relations in Magnetostatics

- Permanent Magnet and Poles (**N** and **S**)
- No isolated Poles
- Magnetic Field lines and Magnetic Field Strength **H**
- Magnetic Flux density **B**
- Permeability of the medium $\mu = \mu_0\, \mu_r$
- Relation between Magnetic field strength and flux density: **B**= μ**H**

AMPERE'S CIRCUIT LAW

Ampere's circuit law states that the line integral of the tangential component of **H** around a *closed* path is the same as the net current I_{enc} enclosed by the path. In other words, the circulation of H equals I_{enc}

$$\oint \mathbf{H} \cdot d\mathbf{l} = I_{\text{enc}}$$

By applying Stoke's theorem,

$$\therefore\ I_{\text{enc}} = \oint_L \mathbf{H} \cdot d\mathbf{l} = \int_S (\nabla \times \mathbf{H}) \cdot d\mathbf{S}$$

Note that:

$$I_{\text{enc}} = \int_S \mathbf{J} \cdot d\mathbf{S}$$

$$\therefore\quad \nabla \times \mathbf{H} = \mathbf{J}$$

$\therefore$ Magnetostatic field is <u>not conservative</u>.

MAGNETIC FLUX DENSITY

The magnetic flux density **B** is similar to the electric flux density **D**.
As **D** = ε**E** in free space, the magnetic flux density **B** is related to the magnetic field intensity **H** according to:

$$\mathbf{B} = \mu_o \mathbf{H}$$

where μ_0 is a constant called *Permeability of free space.* The constant is in henrys/meter (H/m) and has the value of 4π X 10^7 H/m

The magnetic flux through a surface *S* is given by

$$\Psi = \int_S \mathbf{B} \cdot d\mathbf{S}$$

where the magnetic flux Ψ is in Webers (Wb) and the magnetic flux density is in Webers/square meter (Wb/m^2) or Teslas.

- Unlike electric flux lines, magnetic flux lines always close upon themselves.
- This is due to the fact that it is not possible to have *isolated magnetic poles (or magnetic charges).*
- For example, if we desire to have an isolated magnetic pole by dividing a magnetic bar successively into two, we end up with pieces each having north and south poles
- We find it impossible to separate the north pole from the south pole.
- An **isolated magnetic** charge does not exist.
- Thus the total flux through a closed surface in a magnetic field must be zero; i.e., $\oint \mathbf{B} \cdot d\mathbf{S} = 0$

$$\oint \mathbf{B} \cdot d\mathbf{S} = 0$$

- This equation is referred to as the law of conservation of magnetic flux or Gauss's law for Magnetostatic fields.
- Although the Magnetostatic field is not conservative, Magnetic flux is conserved.
- By applying the divergence theorem:

$$\therefore \quad \oint_S \mathbf{B} \cdot d\mathbf{S} = \int_v \nabla \cdot \mathbf{B}\, dv = 0$$

$$\therefore \quad \nabla \cdot \mathbf{B} = 0$$

Applied Electromagnetic Theory

EC303 Module – 2

Maxwell's Equation from fundamental laws.

Points to remember….

- Through out the lecture, following notations will be used…
- Scalars are represented in plain form i.e. V
- Vectors will be represented by bold type phase i.e., **D**, **J**, **B**, **E** etc.
- Use of line integrals, contour or surface integrals and volume integrals will use single integral sign with bottom script stating type.

INTRODUCTION

- Electrostatic fields denoted by **E**(x, y, z) and Magnetostatic fields represented by **H**(x, y, z).
- These are static, or time invariant, EM fields.
- There is also Electric and magnetic fields that are dynamic, or time varying.
- In static EM fields, electric and magnetic fields are independent of each other whereas in dynamic EM fields, the two fields are interdependent.
- A time-varying electric field necessarily involves a corresponding time-varying magnetic field.
- A time-varying EM fields, represented by **E**(x, y, z, t) and **H**(x, y, z, t), are of more practical value than static EM fields.
- Electrostatic fields are usually produced by static electric charges whereas Magnetostatic fields are due to motion of electric charges with uniform velocity (direct current) or static magnetic charges (magnetic poles).
- Time-varying fields or waves are usually due to accelerated charges or time-varying currents.
- Any pulsating current will produce radiation (time-varying fields).

stationary charges → electrostatic fields

steady currents → magnetostatic fields

time-varying currents → electromagnetic fields (or waves)

FARADAY'S LAW

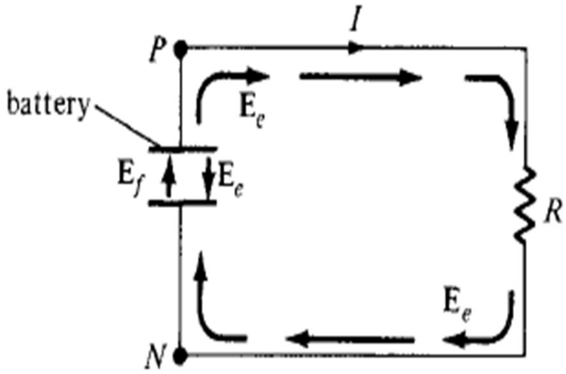
Faraday discovered that the **induced emf,** V_{emf} (in volts), in any closed circuit is equal to the time rate of change of the magnetic flux linkage by the circuit.

$$\therefore\ V_{emf} = -\frac{d\lambda}{dt} = -N\frac{d\Psi}{dt}$$

- where *N* is the number of turns in the circuit and V is the flux through each turn.

Lenz's law

- The induced voltage acts in such a way as to oppose the flux producing it.
- The direction of current flow in the circuit is such that the induced magnetic field produced by the induced current will oppose the original magnetic field.



A circuit showing emf-producing field and electrostatic field

where the battery is a source of emf.
The electrochemical action of the battery results in an emf-produced field E_f.
Due to the accumulation of charge at the battery terminals, an electrostatic field $\mathbf{E}_e$ also exists.

- The total electric field at any point is:
- $\mathbf{E} = \mathbf{E}_f + \mathbf{E}_e$
- Note that E_f is zero outside the battery, $\mathbf{E}_f$ and *Ee* have opposite directions in the battery, and the direction of E_e inside the battery is opposite to that outside it.

$$\therefore \oint_L \mathbf{E} \cdot d\mathbf{l} = \oint_L \mathbf{E}_f \cdot d\mathbf{l} + 0 = \int_N^P \mathbf{E}_f \cdot d\mathbf{l} \qquad \text{(through battery)}$$

$$\therefore V_{emf} = \int_N^P \mathbf{E}_f \cdot d\mathbf{l}$$

because $\mathbf{E}_e$ is conservative.

- For a circuit with a single turn *(N* = 1),

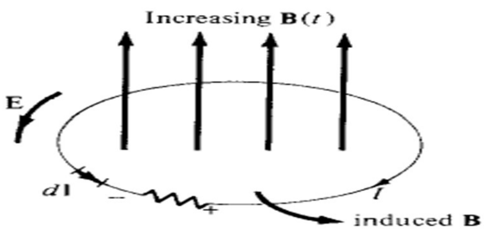
$$V_{emf} = -\frac{d\Psi}{dt}$$

$$\therefore\ V_{emf} = \oint_L \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt}\int_S \mathbf{B} \cdot d\mathbf{S}$$

- The variation of flux caused in three ways:
  - By having a stationary loop in a time-varying **B** field
  - By having a time-varying loop area in a static **B** field
  - By having a time-varying loop area in a time-varying **B** field.

Stationary loop in a time-varying B field

- Consider a stationary conducting loop is in a time varying magnetic **B** field.



Induced emf due to a stationary loop in a time varying B field.

$$\therefore \quad V_{\text{emf}} = \oint_L \mathbf{E} \cdot d\mathbf{l} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S}$$

By applying **Stokes's theorem**:

$$\therefore \quad \int_S (\nabla \times \mathbf{E}) \cdot d\mathbf{S} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S}$$

$$\therefore \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

- This is one of the Maxwell's equations for time-varying fields. It shows that the time varying E field is not conservative i.e., not equal to zero.
- The work done in taking a charge about a closed path in a time-varying electric field, for example, is due to the energy from the time-varying magnetic field.

Points to Ponder ….

- The term **J** represents the conduction current.
- The term **D** is the displacement current.
- The term $\mathbf{J}_d$ = d**D**/dt is known as *Displacement Current Density*
- What is ρ_v ? Is it a scalar or vector?
- What is **B** ?
- What is **H** ? How it relates to **B** ?
- What is **E** ? How it relates to **D** ?

Ampere's circuital law

- Ampere's circuital law states that line integral of tangential component of H around a closed path is same as the net current I_{enc} enclosed by the path.
- Reconsider Ampere's curl equation for magnetic fields for time-varying conditions.
- For static EM fields, $\nabla \times \mathbf{H} = \mathbf{J}$
- But the divergence of the curl of any vector field is identically zero,

$$\therefore \quad \nabla \cdot (\nabla \times \mathbf{H}) = 0 = \nabla \cdot \mathbf{J}$$

- The continuity of current however, requires that,

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho_v}{\partial t} \neq 0$$

- Alas! We have a dilemma due to contradiction!
- Need to modify and add a term to main so that it becomes:

$$\nabla \times \mathbf{H} = \mathbf{J} + \mathbf{J}_d$$

- Again, the divergence of the curl of any vector is zero.

$$\therefore \quad \nabla \cdot (\nabla \times \mathbf{H}) = 0 = \nabla \cdot \mathbf{J} + \nabla \cdot \mathbf{J}_d$$

$$\nabla \cdot \mathbf{J}_d = -\nabla \cdot \mathbf{J} = \frac{\partial \rho_v}{\partial t} = \frac{\partial}{\partial t}(\nabla \cdot \mathbf{D}) = \nabla \cdot \frac{\partial \mathbf{D}}{\partial t}$$

$$\therefore \quad \mathbf{J}_d = \frac{\partial \mathbf{D}}{dt}$$

$$\therefore \quad \nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

This is Maxwell's 2nd equation (based on Ampere's circuit law) for a time-varying field.

The term $\mathbf{J}_d$ = d**D**/dt is known as *Displacement Current Density*

Maxwell's equations in differential and integral form from modified form of Ampere's circuital law

- James clerk Maxwell is regarded as the founder of electromagnetic theory in present form.
- Maxwell's celebrated work led to the discovery of Electromagnetism that Maxwell put together in form of four equations.
- The integral form of Maxwell's equations depicts the underlying physical laws.
- Whereas the differential form is used in solving problems.
- For a field qualified as electromagnetic field, it must satisfy all four Maxwell's equations.

TABLE 9.1 Generalized Forms of Maxwell's Equations

Differential Form	Integral Form	Remarks
$\nabla \cdot \mathbf{D} = \rho_v$	$\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_v \rho_v \, dv$	Gauss's law
$\nabla \cdot \mathbf{B} = 0$	$\oint_S \mathbf{B} \cdot d\mathbf{S} = 0$	Nonexistence of isolated magnetic charge*
$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$	$\oint_L \mathbf{E} \cdot d\mathbf{l} = -\frac{\partial}{\partial t} \int_S \mathbf{B} \cdot d\mathbf{S}$	Faraday's law
$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$	$\oint_L \mathbf{H} \cdot d\mathbf{l} = \int_S \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}\right) \cdot d\mathbf{S}$	Ampere's circuit law

*This is also referred to as Gauss's law for magnetic fields.

Also the **Equation of continuity**:

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho_v}{\partial t}$$

The **Constitutive relations**

$$\mathbf{D} = \varepsilon \mathbf{E}$$

$$\mathbf{B} = \mu \mathbf{H}$$

$$\mathbf{J} = \sigma \mathbf{E}$$

TABLE 9.2 Time-Harmonic Maxwell's Equations Assuming Time Factor $e^{j\omega t}$

Point Form	Integral Form
$\nabla \cdot \mathbf{D}_s = \rho_{vs}$	$\oint \mathbf{D}_s \cdot d\mathbf{S} = \int \rho_{vs}\, dv$
$\nabla \cdot \mathbf{B}_s = 0$	$\oint \mathbf{B}_s \cdot d\mathbf{S} = 0$
$\nabla \times \mathbf{E}_s = -j\omega \mathbf{B}_s$	$\oint \mathbf{E}_s \cdot d\mathbf{l} = -j\omega \int \mathbf{B}_s \cdot d\mathbf{S}$
$\nabla \times \mathbf{H}_s = \mathbf{J}_s + j\omega \mathbf{D}_s$	$\oint \mathbf{H}_s \cdot d\mathbf{l} = \int (\mathbf{J}_s + j\omega \mathbf{D}_s) \cdot d\mathbf{S}$

Boundary Conditions

- There is commonly the existence of the electric field in a homogeneous medium.
- If the field exists in a region consisting of two different media, the conditions that the field must satisfy at the interface separating the media are called **Boundary conditions**.
- These conditions are helpful in determining the field on one side of the boundary if the field on the other side is known.
- The conditions will be dictated by the types of material the media are made of.

- Consider the boundary conditions at an interface separating:
  - dielectric (ε_{r1}) and dielectric (ε_{r2})
  - conductor and dielectric
  - conductor and free space
- To determine the boundary conditions, we need to use Maxwell's equations:

$$\oint \mathbf{E} \cdot d\mathbf{l} = 0 \qquad \oint \mathbf{D} \cdot d\mathbf{S} = Q_{enc}$$

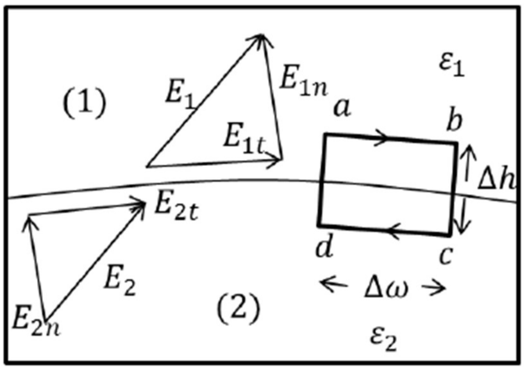
- Also we need to decompose the electric field intensity **E** into two orthogonal components:

$$\mathbf{E} = \mathbf{E}_t + \mathbf{E}_n$$

$\mathbf{E}_t$ →tangential component of E
$\mathbf{E}_n$ →normal component of E to the interface of interest.

Dielectric-Dielectric Boundary Conditions

- Consider the **E** field existing in a region consisting of two different dielectrics characterized by $\varepsilon_1 = \varepsilon_0 \varepsilon_{r1}$ and $\varepsilon_2 = \varepsilon_0 \varepsilon_{r2}$



$$\mathbf{E}_1 = \mathbf{E}_{1t} + \mathbf{E}_{1n}$$

$$\mathbf{E}_2 = \mathbf{E}_{2t} + \mathbf{E}_{2n}$$

We apply eq. 1 to the closed path *abcda* assuming that the path I very small w.r.t., the variation of **E**.

$$0 = E_{1t}\,\Delta w - E_{1n}\frac{\Delta h}{2} - E_{2n}\frac{\Delta h}{2} - E_{2t}\,\Delta w + E_{2n}\frac{\Delta h}{2} + E_{1n}\frac{\Delta h}{2}$$

where $E_t = |\mathbf{E}_t|$ and $E_n = |\mathbf{E}_n|$.

As $\Delta h \to 0$, $\quad E_{1t} = E_{2t}$

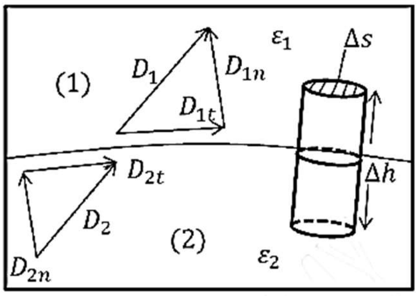
Thus the tangential components of **E** are the same on the two sides of the boundary.
So **E**, undergoes no change on the boundary and it is said to be *Continuous* across the boundary. $\frac{D_{1t}}{\varepsilon_1} = E_{1t} = E_{2t} = \frac{D_{2t}}{\varepsilon_2}$

$$\therefore \quad \frac{D_{1t}}{\varepsilon_1} = \frac{D_{2t}}{\varepsilon_2}$$

Since **D** = ε**E** = $\mathbf{D}_t + \mathbf{D}_n$

$$\frac{D_{1t}}{\varepsilon_1} = E_{1t} = E_{2t} = \frac{D_{2t}}{\varepsilon_2}$$

$$\therefore \quad \frac{D_{1t}}{\varepsilon_1} = \frac{D_{2t}}{\varepsilon_2}$$



i.e., D_t undergoes some change across the interface.
Hence D_t is said to be *Discontinuous* across the interface.

Similarly, we apply eq. (2) to the pillbox (Gaussian surface) of Figure (b). Allowing Δh —> 0 gives,

$$\Delta Q = \rho_S \Delta S = D_{1n} \Delta S - D_{2n} \Delta S$$

$$\therefore \; D_{1n} - D_{2n} = \rho_S$$

where ρ_s is the free charge density placed deliberately at the boundary.

This is based on the assumption that **D** is directed from region 2 to region 1.

If no free charges exist at the interface (i.e., charges are not deliberately placed there), $\rho_s = 0$

$$\therefore \; D_{1n} = D_{2n}$$

Thus the normal component of **D** is continuous across the interface; i.e., D_n undergoes no change at boundary.

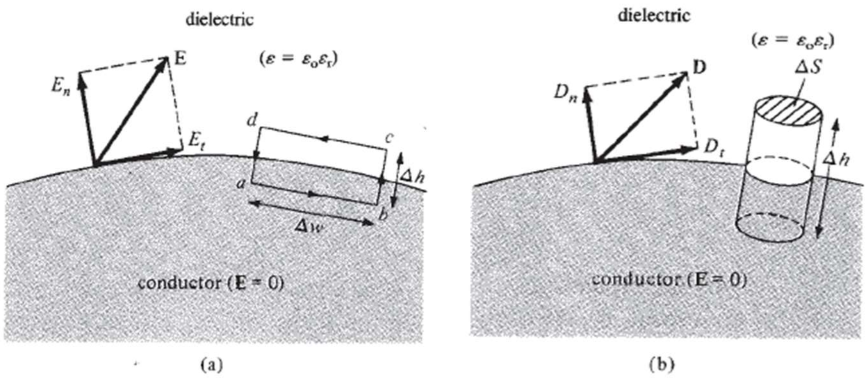
Since **D** = ε**E**, $\therefore \; \varepsilon_1 E_{1n} = \varepsilon_2 E_{2n}$

shows that the normal component of **E** is discontinuous at the boundary.

Equations are collectively referred to as *Boundary conditions.*

They must be satisfied by an electric field at the boundary separating two different dielectrics.

Conductor-Dielectric Boundary Conditions



Conductor-dielectric boundary

- The conductor is assumed to be perfect (σ--> ∞ or ρ_c--> 0).
- Although such a conductor is not practically realizable, regard conductors such as copper and silver as though they were perfect conductors.
- To determine the boundary conditions for a conductor-dielectric interface, follow the same procedure used for dielectric-dielectric interface except that **E** = 0 inside the conductor.
- Applying) to the closed path *abcda* of gives:

$$\oint \mathbf{E} \cdot d\mathbf{l} = 0 \qquad \oint \mathbf{D} \cdot d\mathbf{S} = Q_{enc}$$

$$0 = 0 \cdot \Delta w + 0 \cdot \frac{\Delta h}{2} + E_n \cdot \frac{\Delta h}{2} - E_t \cdot \Delta w - E_n \cdot \frac{\Delta h}{2} - 0 \cdot \frac{\Delta h}{2}$$

As $\Delta h \to 0$, $E_t = 0$

Similarly, by applying second eqn to the pillbox of Figure (b) and letting $\Delta h \to 0$,

$$\Delta Q = D_n \cdot \Delta S - 0 \cdot \Delta S$$

$$\therefore \; D_n = \frac{\Delta Q}{\Delta S} = \rho_S$$

$$\therefore \; D_n = \rho_S$$

Thus under static conditions, the following conclusions can be made about a perfect conductor:

1. No electric field may exist *within* a conductor; that is,

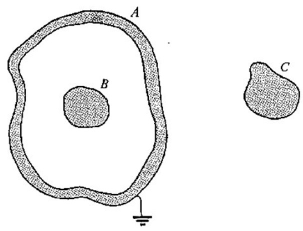
$$\rho_v = 0, \qquad \mathbf{E} = 0$$

2. Since **E**= **-** ∇V = 0, there can be no potential difference between any two points in the conductor; i.e., a conductor is an equipotential body.
3. The electric field **E** can be external to the conductor and *normal* to its surface; i.e.,

$$D_t = \varepsilon_o \varepsilon_r E_t = 0, \qquad D_n = \varepsilon_o \varepsilon_r E_n = \rho_S$$

Electrostatic Screening or Shielding

- An important application of the fact that **E** = 0 inside a conductor is in *electrostatic screening* or *shielding.*
- If conductor *A* kept at zero potential surrounds conductor *B* as shown, *B* is said to be electrically screened by *A* from other electric systems, such as conductor C, outside *A.*
- Similarly, conductor *C* outside *A* is screened by *A* from *B.*



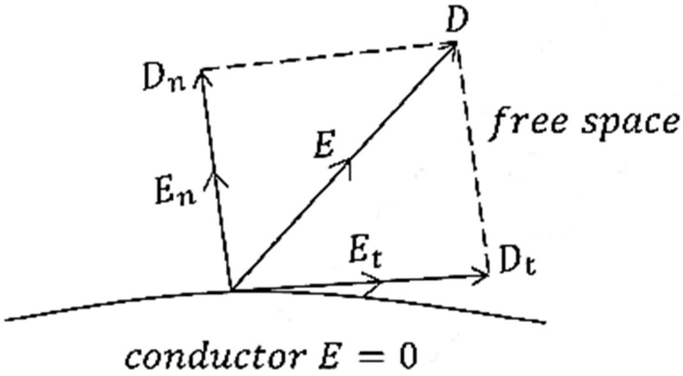
- Thus conductor A acts like a screen or shield and the electrical conditions inside and outside the screen are completely independent of each other.

Conductor-Free Space Boundary Conditions

- This is a special case of the conductor-dielectric conditions.
- The boundary conditions at the interface between a conductor and free space can be obtained by replacing ε_r by 1.
- the electric field E is external to the conductor and normal to its surface.
- Thus the boundary conditions are:

$$D_t = \varepsilon_0 E_t = 0, \qquad D_n = \varepsilon_0 E_n = \rho_S$$

- It should be noted again that **E** field must approach a conducting surface normally.



Conductor-Free Space Interface

Magnetic Boundary Conditions

- Define magnetic boundary conditions as the conditions that **H** or **B** must satisfy at the boundary between two different media.
- Use Gauss's law for magnetic fields.

$$\oint \mathbf{B} \cdot d\mathbf{S} = 0$$

- and Ampere's circuit law,

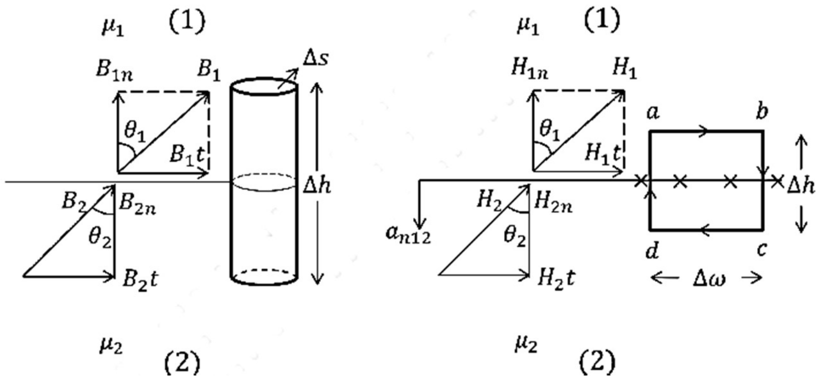
$$\oint \mathbf{H} \cdot d\mathbf{l} = I$$

- Consider the boundary between two magnetic media 1 and 2, characterized, respectively, by μ_1 and μ_1 as shown.
- Applying eq. (1) to the pillbox (Gaussian surface) of Figure (a) and allowing Δh —> 0, we get,

$$B_{1n}\,\Delta S - B_{2n}\,\Delta S = 0$$

$$\mathbf{B}_{1n} = \mathbf{B}_{2n} \qquad \mu_1\mathbf{H}_{1n} = \mu_2\mathbf{H}_{2n}$$

- This shows that the normal component of B is continuous at the boundary.
- It also shows that the normal component of **H** is discontinuous at the boundary;
- **H** undergoes some change at the interface.



Boundary conditions between two magnetic media: (a) for **B**, (b) for **H.**

Similarly, apply Ampere’s law eq. (b) to the closed path *abcda* of Fig b where surface current *K* on the boundary is assumed normal to the path.
This obtains,

$$K \cdot \Delta w = H_{1t} \cdot \Delta w + H_{1n} \cdot \frac{\Delta h}{2} + H_{2n} \cdot \frac{\Delta h}{2} - H_{2t} \cdot \Delta w - H_{2n} \cdot \frac{\Delta h}{2} - H_{1n} \cdot \frac{\Delta h}{2}$$

As Δh --> 0, eq leads to, $H_{1t} - H_{2t} = K$

This shows that the tangential component of **H** is also discontinuous.

$$\frac{B_{1t}}{\mu_1} - \frac{B_{2t}}{\mu_2} = K$$

In the general case, becomes,

$$(\mathbf{H}_1 - \mathbf{H}_2) \times \mathbf{a}_{n12} = \mathbf{K}$$

where $\mathbf{a}_{n12}$ is a unit vector normal to the interface and is directed from medium 1 to medium 2.
If the boundary is free of current or the media are not conductors (for *K* is free current density), *K* = 0,

$$\mathbf{H}_{1t} = \mathbf{H}_{2t} \qquad \frac{\mathbf{B}_{1t}}{\mu_1} = \frac{\mathbf{B}_{2t}}{\mu_2}$$

Thus the tangential component of **H** is continuous while that of **B** is discontinuous at the boundary.

If the fields make an angle θ with the normal to the interface,

$$B_1 \cos \theta_1 = B_{1n} = B_{2n} = B_2 \cos \theta_2$$ ....(1)

$$\frac{B_1}{\mu_1} \sin \theta_1 = H_{1t} = H_{2t} = \frac{B_2}{\mu_2} \sin \theta_2$$ ....(2)

eqn(2)/eqn(1) ....

$$\frac{\tan \theta_1}{\tan \theta_2} = \frac{\mu_1}{\mu_2}$$

Law of refraction of Mag flux lines at a boundary with no surface current

Solution of Wave Equations

- Solve Maxwell's equations and derive EM wave motion in the following media:

1. Free space ($\sigma = 0, \varepsilon = \varepsilon_o, \mu = \mu_o$)
2. Lossless dielectrics ($\sigma = 0, \varepsilon = \varepsilon_r\varepsilon_o, \mu = \mu_r\mu_o$, or $\sigma \ll \omega\varepsilon$)
3. Lossy dielectrics ($\sigma \neq 0, \varepsilon = \varepsilon_r\varepsilon_o, \mu = \mu_r\mu_o$)
4. Good conductors ($\sigma \simeq \infty, \varepsilon = \varepsilon_o, \mu = \mu_r\mu_o$, or $\sigma \gg \omega\varepsilon$)

Wave Propagation in Lossy Dielectrics

- A lossy **dielectric** is a medium in which an EM wave loses power as it propagates due to poor conduction.
- A lossy dielectric is a partially conducting medium (imperfect dielectric or imperfect conductor) with $\sigma \neq 0$,
- A lossless dielectric (perfect or good dielectric) is one in which $\sigma = 0$.
- Consider a linear, isotropic, homogeneous, lossy dielectric medium that is charge free ($\rho_v = 0$).
- Maxwell's equations become: (quashing time factor $e^{j\omega t}$)

$$\nabla \cdot \mathbf{E}_s = 0 \quad \text{................(1)}$$

$$\nabla \cdot \mathbf{H}_s = 0 \quad \text{................(2)}$$

$$\nabla \times \mathbf{E}_s = -j\omega\mu\mathbf{H}_s \quad \text{................(3)}$$

$$\nabla \times \mathbf{H}_s = (\sigma + j\omega\varepsilon)\mathbf{E}_s \quad \text{................(4)}$$

Taking the curl of both sides of eq. (3) gives:

$$\nabla \times \nabla \times \mathbf{E}_s = -j\omega\mu \, \nabla \times \mathbf{H}_s \quad \text{................(5)}$$

Applying the vector identity

$$\nabla \times \nabla \times \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A}$$

to LHS of eq. (5) and invoking eqs. (1) and (4), we obtain,

$$\nabla(\underbrace{\nabla \cdot \mathbf{E}_s}_{0}) - \nabla^2\mathbf{E}_s = -j\omega\mu(\sigma + j\omega\varepsilon)\mathbf{E}_s$$

$$\nabla^2\mathbf{E}_s - \gamma^2\mathbf{E}_s = 0 \quad \text{................(6)}$$

where $\gamma^2 = j\omega\mu(\sigma + j\omega\varepsilon)$ ....(7)

γ is called the Propagation constant of the medium.

- By a similar procedure, it can be shown that for the **H** field,

$$\nabla^2 \mathbf{H}_s - \gamma^2 \mathbf{H}_s = 0 \quad \text{.................}(8)$$

- Equations (6) and (8) are known as homogeneous vector <u>Helmholtz 's equations</u> or simply vector <u>Wave equations</u>.
- In Cartesian coordinates, eq. (6), is equivalent to three scalar wave equations, one for each component of **E** along $\mathbf{a}_x$, $\mathbf{a}_y$, and $\mathbf{a}_z$.
- Since *y* in eqs. (6) to (8) is a complex quantity, we may let,

$$\gamma = \alpha + j\beta \quad \text{.................}(9)$$

- From eqns (7) and (9),

$$-\mathrm{Re}\,\gamma^2 = \beta^2 - \alpha^2 = \omega^2\mu\varepsilon \quad \text{.................}(10)$$

$$|\gamma^2| = \beta^2 + \alpha^2 = \omega\mu\sqrt{\sigma^2 + \omega^2\varepsilon^2} \quad \text{.................}(11)$$

- From eqs. (10) and (11), we obtain,

$$\alpha = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1 + \left[\frac{\sigma}{\omega\varepsilon}\right]^2} - 1\right]} \quad \text{.................}(12)$$

$$\beta = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1 + \left[\frac{\sigma}{\omega\varepsilon}\right]^2} + 1\right]} \quad \text{.................}(13)$$

Assume that the wave propagates along $+\mathbf{a}_z$ and that $\mathbf{E}_s$ has only an x-component,

$$\mathbf{E}_s = E_{xs}(z)\mathbf{a}_x \qquad \text{................(14)}$$

Substitute this into eq. (6) yields,

$$(\nabla^2 - \gamma^2)E_{xs}(z) = 0 \qquad \text{................(15)}$$

$$\underbrace{\frac{\partial^2 E_{xs}(z)}{\partial x^2}}_{0} + \underbrace{\frac{\partial^2 E_{xs}(z)}{\partial y^2}}_{0} + \frac{\partial^2 E_{xs}(z)}{\partial z^2} - \gamma^2 E_{xs}(z) = 0$$

$$\left[\frac{d^2}{dz^2} - \gamma^2\right]E_{xs}(z) = 0 \qquad \text{................(16)}$$

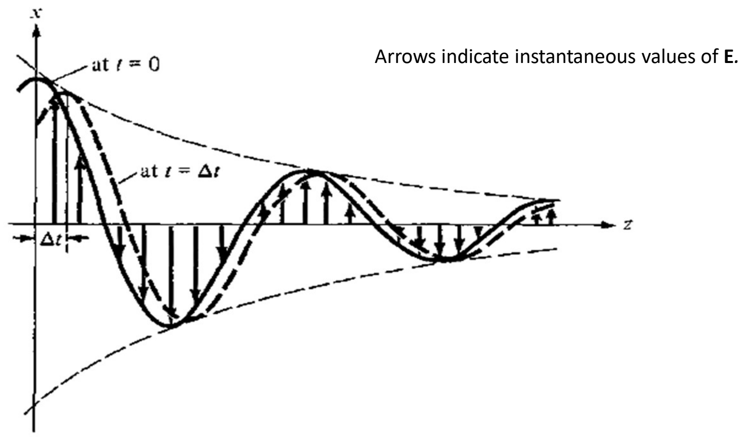
This is a scalar wave equation, a linear homogeneous differential equation, with solution,

$$E_{xs}(z) = E_o e^{-\gamma z} + E'_o e^{\gamma z} \qquad \text{................(17)}$$

where E_o and E_o' are constants.
The field must be finite at infinity requires that $E_o' = 0$.
Since, $e^{\gamma z}$ denotes a wave traveling along $-\mathbf{a_z}$ whereas we assume wave propagation along $\mathbf{a_z}$, $E_o' = 0$.
Whichever way we look at it, $E_o' = 0$.
Inserting the time factor $e^{j\omega t}$ into eq. (17) and using eq. (9), we obtain,

$$\mathbf{E}(z, t) = \text{Re}\,[E_{xs}(z)e^{j\omega t}\mathbf{a}_x] = \text{Re}\,(E_o e^{-\alpha z}e^{j(\omega t-\beta z)}\mathbf{a}_x) \qquad \text{................(18)}$$

$$\mathbf{E}(z, t) = E_o e^{-\alpha z}\cos(\omega t - \beta z)\mathbf{a}_x \qquad \text{................(19)}$$



E-field with x-component traveling along +z-direction at times $t = 0$ and $t = \Delta t$;

$$\mathbf{H}(z, t) = \text{Re}\,(H_o e^{-\alpha z} e^{j(\omega t - \beta z)}\,\mathbf{a}_y) \quad \text{................(19)}$$

$$H_o = \frac{E_o}{\eta} \quad \text{................(20)}$$

and η is a complex quantity known as the *intrinsic impedance* (in ohms) of the medium.

$$\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}} = |\eta|\angle\theta_\eta = |\eta|e^{j\theta_\eta} \quad \text{................(21)}$$

$$|\eta| = \frac{\sqrt{\mu/\varepsilon}}{\left[1 + \left(\frac{\sigma}{\omega\varepsilon}\right)^2\right]^{1/4}}, \qquad \tan 2\theta_\eta = \frac{\sigma}{\omega\varepsilon} \quad \text{................(22)}$$

$$0 \le \theta_\eta \le 45°$$

Substituting eqs. (19) and (20) into eq. (21) gives,

$$\mathbf{H} = \text{Re}\left[\frac{E_o}{|\eta|e^{j\theta_\eta}}\,e^{-\alpha z}e^{j(\omega t - \beta z)}\,\mathbf{a}_y\right]$$

$$\mathbf{H} = \frac{E_o}{|\eta|}\,e^{-\alpha z}\cos(\omega t - \beta z - \theta_\eta)\,\mathbf{a}_y \quad \text{................(23)}$$

From eqs. (19) and (23) as the wave propagates along $\mathbf{a}_z$, it decreases or attenuates in amplitude by a factor e $^{-\alpha z}$, and hence α is known as the Attenuation constant or Attenuation factor of the medium.

It is a measure of the spatial rate of decay of the wave in the medium, measured in Nepers per meter (Np/m) or in decibels per meter (dB/m).
An attenuation of 1 Neper denotes a reduction to e^{-1} of the original value whereas an increase of 1 Neper indicates an increase by a factor of *e.*

$$1 \text{ Np} = 20 \log_{10} e = 8.686 \text{ dB} \quad \text{……………(23)}$$

If α = 0, as is the case for a lossless medium and free space, α = 0 and the wave is not attenuated as it propagates.
The quantity β is a measure of the phase shift per length and is called the *phase constant* or *wave number.*
In terms of β, the wave velocity u and wavelength λ are given by,

$$u = \frac{\omega}{\beta}, \qquad \lambda = \frac{2\pi}{\beta} \quad \text{……………(24)}$$

E and **H** are out of phase by θ_η , at any instant of time due to the complex intrinsic impedance of the medium. Thus at any time, **E** leads **H** (or **H** lags **E**) by θ_η

The ratio of the magnitude of the conduction current density **J** to that of the displacement current density $\mathbf{J}_d$ in a lossy medium is,

$$\frac{|\mathbf{J}_s|}{|\mathbf{J}_{ds}|} = \frac{|\sigma \mathbf{E}_s|}{|j\omega\varepsilon \mathbf{E}_s|} = \frac{\sigma}{\omega\varepsilon} = \tan\theta$$ …………….(25)

$$\tan\theta = \frac{\sigma}{\omega\varepsilon}$$ …………….(26)

where tan θ is known as the loss tangent and θ is the loss angle of the medium

PLANE WAVES IN LOSSLESS DIELECTRICS

In a lossless dielectric, $\sigma \ll \omega\varepsilon$.

$$\sigma \simeq 0, \qquad \varepsilon = \varepsilon_o\varepsilon_r, \qquad \mu = \mu_o\mu_r$$ …………….(26)

$$\alpha = 0, \qquad \beta = \omega\sqrt{\mu\varepsilon}$$

$$u = \frac{\omega}{\beta} = \frac{1}{\sqrt{\mu\varepsilon}}, \qquad \lambda = \frac{2\pi}{\beta}$$ …………….(27)

$$\eta = \sqrt{\frac{\mu}{\varepsilon}} \angle 0°$$

…………….(28)

and thus **E** and **H** are in time phase with each other.

PLANE WAVES IN FREE SPACE

This is a special case

$$\sigma = 0, \qquad \varepsilon = \varepsilon_o, \qquad \mu = \mu_o \qquad \text{.................(29)}$$

$$\alpha = 0, \qquad \beta = \omega\sqrt{\mu_o\varepsilon_o} = \frac{\omega}{c} \qquad \text{.................(30)}$$

$$u = \frac{1}{\sqrt{\mu_o\varepsilon_o}} = c, \qquad \lambda = \frac{2\pi}{\beta} \qquad \text{.................(31)}$$

where $c = 3 \times 10^8$ m/s, the speed of light in a vacuum.
EM wave travels in free space at the speed of light.
It shows that light is the manifestation of an EM wave.
In other words, light is characteristically Electromagnetic.

PLANE WAVES IN GOOD CONDUCTORS

This is another special case

A perfect, or good conductor, is one in which,

$\sigma \gg \omega\varepsilon$ so that $\sigma/\omega\varepsilon \to \infty$;

$$\sigma \simeq \infty, \qquad \varepsilon = \varepsilon_o, \qquad \mu = \mu_o\mu_r \qquad \text{.................(32)}$$

$$\alpha = \beta = \sqrt{\frac{\omega\mu\sigma}{2}} = \sqrt{\pi f\mu\sigma} \qquad \text{.................(33)}$$

$$u = \frac{\omega}{\beta} = \sqrt{\frac{2\omega}{\mu\sigma}}, \qquad \lambda = \frac{2\pi}{\beta} \qquad \text{.................(34)}$$

$$\eta = \sqrt{\frac{\omega\mu}{\sigma}} \angle 45^\circ \qquad \text{................(35)}$$

and thus **E** leads **H** by 45°.

$$\mathbf{E} = E_o e^{-\alpha z} \cos(\omega t - \beta z)\, \mathbf{a}_x \qquad \text{................(36)}$$

$$\mathbf{H} = \frac{E_o}{\sqrt{\frac{\omega\mu}{\sigma}}} e^{-\alpha z} \cos(\omega t - \beta z - 45^\circ)\, \mathbf{a}_y \qquad \text{................(37)}$$

Therefore, as **E** (or **H**) wave travels in a conducting medium, its amplitude is attenuated factor $e^{-\alpha z}$.

The distance δ, through which the wave amplitude decreases by a factor e^{-1}(about 37%) is called Skin depth or Penetration depth of the medium;

$$E_o e^{-\alpha\delta} = E_o e^{-1} \qquad \text{................(38)}$$

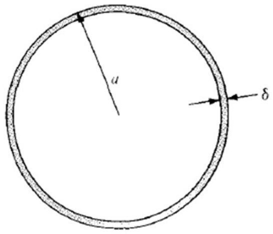
$$\delta = \frac{1}{\alpha} \qquad \text{................(39)}$$

The **skin depth** is a measure of the depth to which an EM wave can penetrate the medium.

Equation is generally valid for any material medium.
For good conductors,

$$\delta = \frac{1}{\sqrt{\pi f \mu \sigma}} \qquad \text{................(40)}$$

- The resistance is called the *dc resistance,* that is,
$$R_{dc} = \frac{\ell}{\sigma S} \qquad \text{................(41)}$$
- The skin depth is useful in calculating the *ac resistance* due to skin effect.



Skin depth at high frequencies $\delta \ll a.$

Define the *surface or skin resistance Rs* (in Ω/m[2]) as the real part of the η for a good conductor.

$$R_s = \frac{1}{\sigma\delta} = \sqrt{\frac{\pi f \mu}{\sigma}} \qquad \text{................(42)}$$

Resistance of a unit width and unit length of conductor. It is equivalent to the dc resistance for a unit length of the conductor having cross-sectional area 1 x δ.
Thus for a given width w and length ***ℓ***, the ac resistance is calculated, assuming a uniform current flow in conductor of thickness δ, i.e.,

$$R_{ac} \simeq \frac{\ell}{\sigma\delta w} = \frac{R_s \ell}{w} \qquad \text{................(43)}$$

where $S \cong \delta w$.
For a conductor wire of radius *a* (see diagram),
w = $2\pi a$, so

$$\frac{R_{ac}}{R_{dc}} = \frac{\dfrac{\ell}{\sigma 2\pi a\delta}}{\dfrac{\ell}{\sigma\pi a^2}} = \frac{a}{2\delta} \qquad \text{................(44)}$$

Since $\delta \ll a$ at high frequencies, this shows that $R_{ac} \gg R_{dc}$.
The ratio of the ac to dc resistance starts at 1.0 for dc and very low frequencies and increases as frequency raises.
The bulk of the current is nonuniformly distributed over a thickness of 5δ of the conductor.
But the power loss is the same as though it were uniformly distributed over a thickness of δ and zero elsewhere.
So δ is clearly referred to as the skin depth.

Applied Electromagnetic Theory

EC303 Module – 3

POWER AND THE POYNTING VECTOR

- Energy can be transported from one point to another point by means of EM waves.
- The rate of such energy transportation can be obtained from Maxwell's equations:

$$\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t} \qquad \text{.......(1)}$$

$$\nabla \times \mathbf{H} = \sigma \mathbf{E} + \varepsilon \frac{\partial \mathbf{E}}{\partial t} \qquad \text{.......(2)}$$

- Apply dot product on both sides of eq. (2) with **E** gives,

$$\therefore \quad \mathbf{E} \cdot (\nabla \times \mathbf{H}) = \sigma E^2 + \mathbf{E} \cdot \varepsilon \frac{\partial \mathbf{E}}{\varepsilon t} \qquad \text{.......(3)}$$

Use the identity,

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B}). \quad ....(4)$$

Letting **A** = **H** and **B** = **E**, in eqn(3)

$$\therefore \ \mathbf{H} \cdot (\nabla \times \mathbf{E}) + \nabla \cdot (\mathbf{H} \times \mathbf{E}) = \sigma E^2 + \mathbf{E} \cdot \varepsilon \frac{\partial \mathbf{E}}{\partial t} \quad ....(5)$$

Use eqn(1),

$$\therefore \ \mathbf{H} \cdot (\nabla \times \mathbf{E}) = \mathbf{H} \cdot \left(-\mu \frac{\partial \mathbf{H}}{\partial t}\right) = -\frac{\mu}{2} \frac{\partial}{\partial t} (\mathbf{H} \cdot \mathbf{H}) \quad ....(6)$$

Work on eqn(5),

$$\therefore \ -\frac{\mu}{2} \frac{\partial H^2}{\partial t} - \nabla \cdot (\mathbf{E} \times \mathbf{H}) = \sigma E^2 + \frac{1}{2} \varepsilon \frac{\partial E^2}{\partial t}$$

Rearranging terms and taking the Volume integral of both sides,

$$\therefore \int_v \nabla \cdot (\mathbf{E} \times \mathbf{H})\, dv = -\frac{\partial}{\partial t} \int_v \left[\frac{1}{2} \varepsilon E^2 + \frac{1}{2} \mu H^2\right] dv - \int_v \sigma E^2\, dv \quad ....(7)$$

Applying the Divergence theorem to the left-hand side gives,

Poynting's theorem

$$\therefore \oint_S (\mathbf{E} \times \mathbf{H}) \cdot d\mathbf{S} = -\frac{\partial}{\partial t} \int_v \left[\frac{1}{2} \varepsilon E^2 + \frac{1}{2} \mu H^2\right] dv - \int_v \sigma E^2\, dv \quad ....(8)$$

↓ Total power leaving the volume = ↓ Rate of decrease in energy stored in electric and magnetic fields − ↓ Ohmic power dissipated

The various terms in the eqn (8) are identified using energy-conservation arguments for EM fields.

1. The first term on the right-hand side of eq. (8) is interpreted as the rate of decrease in Energy stored in the Electric and Magnetic fields.

2. The second term is the Power dissipated due to the fact that the medium is conducting ($\sigma \neq 0$).

3. The quantity **E** x **H** on the left-hand side of eq. (8) is known as the Poynting vector **P** in watts/sq m(W/m2);

$$\therefore \quad \mathbf{P} = \mathbf{E} \times \mathbf{H} \qquad \text{.......(9)}$$

It represents the instantaneous power density vector associated with the EM field at a given point.

The integration of the Poynting vector over any closed surface gives the net power flowing out of that surface.

Poynting's theorem states that the net power flowing out of a given volume *v* is equal to the time rate of decrease in the energy stored within *v* minus the conduction losses.

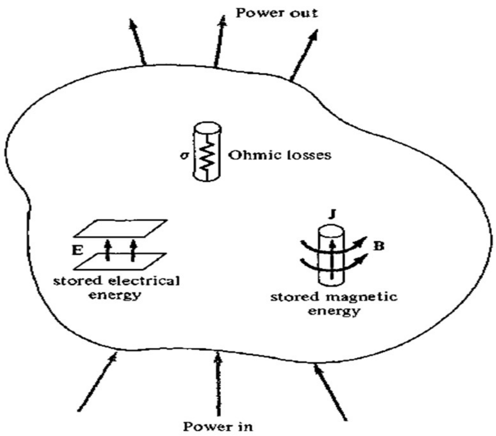


Illustration of power balance for EM fields

It should be noted that **P** is normal to both **E** and **H** and is therefore along the direction of wave propagation $\mathbf{a}_k$ for uniform plane waves.

$$\therefore \quad \mathbf{a}_k = \mathbf{a}_E \times \mathbf{a}_H \quad \text{.......(10)}$$

The fact that **P** points along $\mathbf{a}_k$ causes **P** to be regarded derisively as a "Pointing" vector.

Again, assume that,

$$\mathbf{E}(z, t) = E_o e^{-\alpha z} \cos(\omega t - \beta z)\, \mathbf{a}_x \quad \text{.......(11)}$$

$$\therefore \quad \mathbf{H}(z, t) = \frac{E_o}{|\eta|} e^{-\alpha z} \cos(\omega t - \beta z - \theta_\eta)\, \mathbf{a}_y \quad \text{.......(12)}$$

$$\mathbf{P}(\mathbf{z}, t) = \frac{E_o^2}{|\eta|} e^{-2\alpha z} \cos(\omega t - \beta z) \cos(\omega t - \beta z - \theta_\eta)\, \mathbf{a}_z \quad \text{.......(13)}$$

$$\mathbf{P}(\mathbf{z}, t) = \frac{E_o^2}{2|\eta|} e^{-2\alpha z} [\cos\theta_\eta + \cos(2\omega t - 2\beta z - \theta_\eta)]\, \mathbf{a}_z \quad \text{.......(14)}$$

To determine the time-average Poynting vector $\mathbf{P}_{ave}(z)$ (in W/m^2), which is of more practical value than the instantaneous Poynting vector $\mathbf{P}(\mathbf{z}, t)$, we integrate eq. (14) over the period $T = 2\pi/\omega$; that is,

$$\therefore \quad \mathbf{P}_{ave}(\mathbf{z}) = \frac{1}{T}\int_0^T \mathbf{P}(\mathbf{z}, t)\,dt \quad \text{.......(15)}$$

This is equivalent to,

$$\therefore \quad \mathbf{P}_{ave}(\mathbf{z}) = \frac{1}{2}\,\mathrm{Re}\,(\mathbf{E}_s \times \mathbf{H}_s^*) \quad \text{.......(16)}$$

Substitute eq. (14) into eq. (15),

$$\therefore\ \mathbf{P}_{ave}(\mathbf{z}) = \frac{E_o^2}{2|\eta|} e^{-2\alpha z} \cos\theta_\eta\, \mathbf{a}_z \quad ....(17)$$

The total time-average Power crossing a given surface **S** is given by ,

$$\therefore\ P_{ave} = \int_S \mathbf{P}_{ave}(\mathbf{z}) \cdot d\mathbf{S} \quad ....(18)$$

Note the various Poynting quantities:

P(x,y,z,t)→ Poynting vector in watts/m and is time varying.

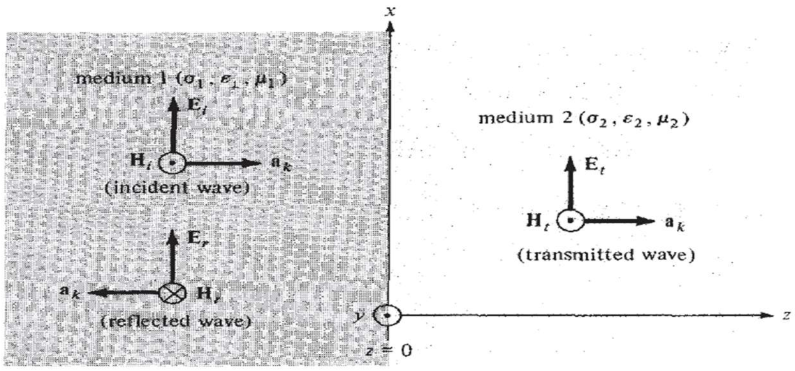
$\mathbf{P}_{ave}$(x,y,z)→ also in watts/meter is the time average of the Poynting vector.
It is a vector but is time invariant.

P_{ave} → is a total time average power thro' a surface in watts and it is a scalar.

REFLECTION OF A PLANE WAVE AT NORMAL INCIDENCE

- Earlier considered uniform plane waves traveling in unbounded, homogeneous media.
- When a plane wave from one medium meets a different medium, it is partly reflected and partly transmitted.
- The proportion of the incident wave that is reflected or transmitted depends on the constitutive parameters (ε, μ, σ) of the two media involved.
- Here assume that the incident wave plane is normal to the boundary between the media

- Suppose that a plane wave propagating along the +z-direction is incident normally on the boundary $z = 0$ between medium 1 ($z < 0$) characterized by ε_1, μ_1, σ_1 and medium 2 ($z > 0$) characterized by ε_2, μ_2, σ_2 as shown.



A plane wave incident normally on an interface between two different media.

Incident Wave

- ($\mathbf{E}_i$, $\mathbf{H}_i$) is traveling along $+a_z$ in medium 1.
- Suppress the time factor $e^{j\omega t}$ and assume that,

$$\mathbf{E}_{is}(z) = E_{io}e^{-\gamma_1 z}\,\mathbf{a}_x \qquad \text{.............(19)}$$

$$\therefore\ \mathbf{H}_{is}(z) = H_{io}e^{-\gamma_1 z}\,\mathbf{a}_y = \frac{E_{io}}{\eta_1}\,e^{-\gamma_1 z}\,\mathbf{a}_y \qquad \text{.............(20)}$$

Reflected Wave

($\mathbf{E}_r$, $\mathbf{H}_r$) is traveling along $-\mathbf{a}_z$ in medium-1.

$$\mathbf{E}_{rs}(z) = E_{ro}e^{\gamma_1 z}\,\mathbf{a}_x \qquad \text{.............(21)}$$

$$\therefore\ \mathbf{H}_{rs}(z) = H_{ro}\,e^{\gamma_1 z}(-\mathbf{a}_y) = -\frac{E_{ro}}{\eta_1}\,e^{\gamma_1 z}\,\mathbf{a}_y \qquad \text{.............(22)}$$

where $\mathbf{E}_{rs}$ has been assumed to be along $\mathbf{a}_x$

Assume that for normal incident, $\mathbf{E}_i$, $\mathbf{E}_r$, and $\mathbf{E}_t$ have the same Polarization.

Transmitted Wave

($\mathbf{E}_t$, $\mathbf{H}_t$) is traveling along $+\mathbf{a}_z$ in medium-2.

$$\mathbf{E}_{ts}(z) = E_{to}\, e^{-\gamma_2 z}\, \mathbf{a}_x \qquad \text{.............(23)}$$

$$\therefore \quad \mathbf{H}_{ts}(z) = H_{to}\, e^{-\gamma_2 z}\, \mathbf{a}_y = \frac{E_{to}}{\eta_2}\, e^{-\gamma_2 z}\, \mathbf{a}_y \qquad \text{.............(24)}$$

E_{i0}, E_{r0}, and E_{t0} are, respectively, the magnitudes of the incident, reflected, and transmitted electric fields at $z = 0$.

The total field in medium 1 comprises both the incident and reflected fields, whereas medium 2 has only the transmitted field, that is,

$$\therefore \quad \mathbf{E}_1 = \mathbf{E}_i + \mathbf{E}_r, \qquad \mathbf{H}_1 = \mathbf{H}_i + \mathbf{H}_r \qquad \text{.............(25)}$$

$$\therefore \quad \mathbf{E}_2 = \mathbf{E}_t, \qquad \mathbf{H}_2 = \mathbf{H}_t \qquad \text{.............(26)}$$

- At the interface z = 0, the boundary conditions require that the tangential components of **E** and **H** fields must be continuous.
- Since the waves are transverse, **E** and **H** fields are entirely tangential to the interface.

Hence at $z = 0$, $\mathbf{E}_{1\tan} = \mathbf{E}_{2\tan}$ and $\mathbf{H}_{1\tan} = \mathbf{H}_{2\tan}$

$$\mathbf{E}_i(0) + \mathbf{E}_r(0) = \mathbf{E}_t(0) \quad \rightarrow \quad E_{io} + E_{ro} = E_{to}$$ ....(27)

$$\mathbf{H}_i(0) + \mathbf{H}_r(0) = \mathbf{H}_t(0) \quad \rightarrow \quad \frac{1}{\eta_1}(E_{io} - \mathbf{E}_{ro}) = \frac{E_{to}}{\eta_2}$$

........(28)

From eqs. (27) and (28), we obtain

$$\therefore \quad E_{ro} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} E_{io}$$ ....(29)

$$\therefore \quad E_{to} = \frac{2\eta_2}{\eta_2 + \eta_1} E_{io}$$ ....(30)

We now define the *reflection coefficient* Γ and the *transmission coefficient* τ from eqs. (29) and (30) as

$$\therefore \quad \Gamma = \frac{E_{ro}}{E_{io}} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} \qquad \text{........(31)}$$

$$\therefore \quad E_{ro} = \Gamma E_{io} \qquad \text{........(32)}$$

$$\therefore \quad \tau = \frac{E_{to}}{E_{io}} = \frac{2\eta_2}{\eta_2 + \eta_1} \qquad \text{........(33)}$$

$$\therefore \quad E_{to} = \tau E_{io} \qquad \text{........(34)}$$

1. $1 + \Gamma = \tau$
2. Both Γ and τ are dimensionless and may be complex. ....(35)
3. $0 \leq |\Gamma| \leq 1$

- The case considered above is the general case.
- Let us now consider a special case when medium 1 is a perfect dielectric (lossless) and medium 2 is a perfect conductor.
- For this case, $\eta_2 = 0$; hence, $\Gamma = -1$, and $\tau = 0$, showing that the wave is totally reflected.
- This should be expected because fields in a perfect conductor must vanish, so there can be no transmitted wave ($\mathbf{E}_2 = 0$).
- The totally reflected wave combines with the incident wave to form a *standing wave.*
- *A* standing wave "stands" and does not travel.
- It consists of two traveling waves ($\mathbf{E}_i$ and $\mathbf{E}_r$) of equal amplitudes but in opposite directions.

Combining eqs. (19) and (21) gives the standing wave in medium 1 as,

$$\mathbf{E}_{1s} = \mathbf{E}_{is} + \mathbf{E}_{rs} = (E_{io}e^{-\gamma_1 z} + E_{ro}e^{\gamma_1 z})\,\mathbf{a}_x \quad \text{........(36)}$$

$$\Gamma = \frac{E_{ro}}{E_{io}} = -1, \sigma_1 = 0, \alpha_1 = 0, \gamma_1 = j\beta_1$$

$$\therefore \quad \mathbf{E}_{1s} = -E_{io}(e^{j\beta_1 z} - e^{-j\beta_1 z})\,\mathbf{a}_x$$

$$\therefore \quad \mathbf{E}_{1s} = -2jE_{io}\sin\beta_1 z\,\mathbf{a}_x \quad \text{........(37)}$$

$$\therefore \quad \mathbf{E}_1 = \mathrm{Re}\,(\mathbf{E}_{1s}e^{j\omega t})$$

$$\therefore \quad \mathbf{E}_1 = 2E_{io}\sin\beta_1 z\sin\omega t\,\mathbf{a}_x \quad \text{........(38)}$$

By taking similar steps, it can be shown that the magnetic field component of the wave is:

$$\therefore \quad \mathbf{H}_1 = \frac{2E_{io}}{\eta_1} \cos \beta_1 z \cos \omega t \, \mathbf{a}_y \qquad \text{........(39)}$$

When media 1 and 2 are both lossless we have another special case $\sigma_1 = 0 = \sigma_2$. In this case, η_1 and η_2 are real and so are Γ and τ.

CASE A

- If $\eta_2 > \eta_1$, $\Gamma > 0$, there is a standing wave in medium 1 but there is also a transmitted wave in medium 2.
- However, the incident and reflected waves have amplitudes that are not equal in magnitude.
- It can be shown that the maximum values of $|\mathbf{E}_1|$ occur at,

$$-\beta_1 z_{max} = n\pi$$

$$\therefore \; z_{max} = -\frac{n\pi}{\beta_1} = -\frac{n\lambda_1}{2}, \qquad n = 0, 1, 2, \ldots \qquad \text{........(40)}$$

and the minimum values of $|\mathbf{E}_1|$ occur at,

$$-\beta_1 z_{min} = (2n + 1)\frac{\pi}{2}$$

$$\therefore\ z_{min} = -\frac{(2n + 1)\pi}{2\beta_1} = -\frac{(2n + 1)}{4}\lambda_1, \qquad n = 0, 1, 2, \ldots \quad \text{........(40)}$$

CASE B

- If $\eta_2 < \eta_1$, $\Gamma < 0$, the locations of $|\mathbf{E}_1|$ maximum are given by eq. (40),
- Whereas those of $|\mathbf{E}_1|$ minimum are given by eq. (39).

1. $|\mathbf{H}_1|$ minimum occurs whenever there is $|\mathbf{E}_1|$ maximum and vice versa.

2. The transmitted wave in medium 2 is a purely traveling wave and consequently there are no maxima or minima in this region.

The ratio of | $\mathbf{E}_1$ |max to | $\mathbf{E}_1$ |min (or | $\mathbf{H}_1$ |max to | $\mathbf{H}_1$ |min) is called the *Standing-Wave Ratio s;* that is,

$$\therefore \quad s = \frac{|\mathbf{E}_1|_{\max}}{|\mathbf{E}_1|_{\min}} = \frac{|\mathbf{H}_1|_{\max}}{|\mathbf{H}_1|_{\min}} = \frac{1 + |\Gamma|}{1 - |\Gamma|} \qquad \text{........(41)}$$

$$\therefore \quad |\Gamma| = \frac{s - 1}{s + 1} \qquad \text{........(42)}$$

- Since $|\Gamma| \leq 1$, it follows that $1 \leq s \leq \infty$.
- The standing-wave ratio is dimensionless and it is expressed in decibels (dB) as,

$$\therefore \quad s \text{ in dB} = 20 \log_{10} s \qquad \text{........(41)}$$

REFLECTION OF A PLANE WAVE AT OBLIQUE INCIDENCE

- a more general situation than previous.
- To simplify the analysis, we will assume that we are dealing with lossless media.
- It can be shown that a uniform plane wave takes the general form of,

$$\therefore\ \mathbf{E}(\mathbf{r}, t) = \mathbf{E}_o \cos(\mathbf{k} \cdot \mathbf{r} - \omega t) = \mathrm{Re}\,[E_o e^{j(\mathbf{k} \cdot \mathbf{r} - \omega t)}] \quad \text{........(42)}$$

- where r = $x\mathbf{a}_x + y\mathbf{a}_y + z\mathbf{a}_z$ is the radius or position vector and $\mathbf{k} = k_x\mathbf{a}_x + k_y\mathbf{a}_y + k_z\mathbf{a}_z$ is the <u>wave number vector</u> or the <u>propagation vector</u>.
- Vector **k** is always in the direction of wave propagation.
- The magnitude of **k** is related to ω according to the dispersion relation

$$k^2 = k_x^2 + k_y^2 + k_z^2 = \omega^2 \mu \varepsilon \quad \text{........(43)}$$

  - Thus, for lossless media, *k* is essentially the same as β in the previous analysis.

- With the general form of **E** as in eq. (42), **Maxwell's equations** reduce to,

$$\mathbf{k} \times \mathbf{E} = \omega\mu\mathbf{H} \qquad \text{........(44a)}$$

$$\mathbf{k} \times \mathbf{H} = -\omega\varepsilon\mathbf{E} \qquad \text{........(44b)}$$

$$\mathbf{k} \cdot \mathbf{H} = 0 \qquad \text{........(44c)}$$

$$\mathbf{k} \cdot \mathbf{E} = 0 \qquad \text{........(44d)}$$

showing that,
(i) **E**, **H**, and **k** are mutually orthogonal, and
(ii) **E** and **H** lie on the plane

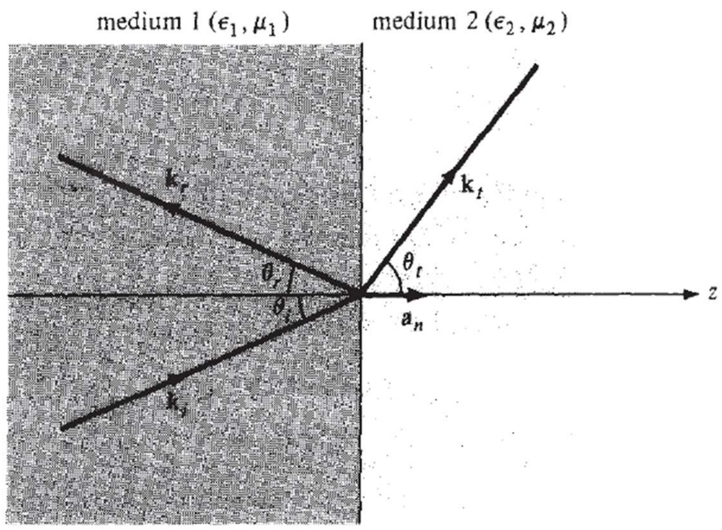
$$\mathbf{k} \cdot \mathbf{r} = k_x x + k_y y + k_z z = \text{constant}$$

From eq. (44a), the **H** field corresponding to the E field in eq. (42) is,

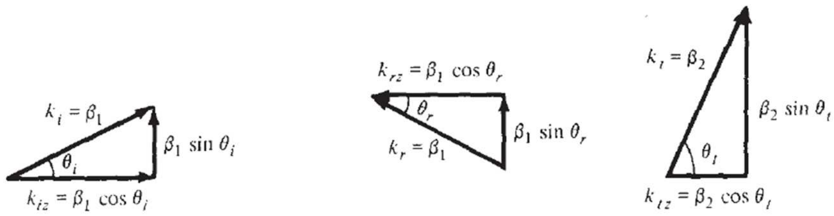
$$\mathbf{H} = \frac{1}{\omega\mu}\mathbf{k} \times \mathbf{E} = \frac{\mathbf{a}_k \times \mathbf{E}}{\eta} \qquad \text{........(45)}$$

Having expressed E and **H** , now consider the oblique incidence of a uniform plane wave at a plane boundary as shown.

- The plane defined by the propagation vector **k** and a unit normal vector $\mathbf{a}_n$ to the boundary is called the *plane of incidence.*
- The angle θ, between **k** and $\mathbf{a}_n$ is the *angle of incidence.*
- Again, both the incident and the reflected waves are in medium 1 while the transmitted or refracted wave is in medium 2.



Oblique incidence of a plane wave:



(b)

illustration of the normal and tangential components of **k**.

Let

$$\mathbf{E}_i = \mathbf{E}_{io} \cos(k_{ix}x + k_{iy}y + k_{iz}z - \omega_i t)$$

$$\mathbf{E}_r = \mathbf{E}_{ro} \cos(k_{rx}x + k_{ry}y + k_{rz}z - \omega_r t) \quad \text{........(46)}$$

$$\mathbf{E}_t = \mathbf{E}_{to} \cos(k_{tx}x + k_{ty}y + k_{tz}z - \omega_t t)$$

- where $\mathbf{k}_i$, $\mathbf{k}_r$, and $\mathbf{k}_t$, with their normal and tangential components are shown in Fig. (b).
- Since the tangential component of **E** must be continuous at the boundary $z = 0$

$$\mathbf{E}_i(z = 0) + \mathbf{E}_r(z = 0) = \mathbf{E}_t(z = 0) \quad \text{........(47)}$$

The only way this boundary condition will be satisfied by the waves in eq. (46) for all *x* and *y* is that,

$$1.\ \omega_i = \omega_r = \omega_t = \omega$$
$$2.\ k_{ix} = k_{rx} = k_{tx} = k_x$$
$$3.\ k_{iy} = k_{ry} = k_{ty} = k_y$$

- Condition 1 implies that the frequency is unchanged.
- Conditions 2 and 3 require that the tangential components of the propagation vectors be continuous (called **Phase matching conditions**).

- This means that the propagation vectors $\mathbf{k}_i$, $\mathbf{k}_r$, and $\mathbf{k}_t$, must all lie in the *plane of incidence.*
- Thus, by conditions 2 and 3,

$$k_i \sin\theta_i = k_r \sin\theta_r \qquad \text{........(48)}$$

$$k_i \sin\theta_i = k_t \sin\theta_t \qquad \text{........(49)}$$

where θ_r is *the angle of reflection* and θ_t is the *angle of transmission.*

But for lossless media,

$$k_i = k_r = \beta_1 = \omega\sqrt{\mu_1\varepsilon_1} \qquad \text{........(50)}$$

$$k_t = \beta_2 = \omega\sqrt{\mu_2\varepsilon_2}$$

From eqs. (48) and (50a), it is clear that,

$$\theta_r = \theta_i \qquad \text{........(51)}$$

so that the angle of reflection θ_r equals the angle of incidence θ_i as in optics.
Also from eqs. (49) and (50),

$$\frac{\sin\theta_t}{\sin\theta_i} = \frac{k_i}{k_t} = \frac{\mu_2}{\mu_1} = \sqrt{\frac{\mu_1\varepsilon_1}{\mu_2\varepsilon_2}} \qquad \text{........(52)}$$

- where $u = \omega/k$ is the phase velocity.
- Equation (52) is the well-known **Snell's law** *,* which can be written as,

$$n_1 \sin\theta_i = n_2 \sin\theta_t \qquad \text{........(53)}$$

$$n_1 = c\sqrt{\mu_1\varepsilon_1} = c/u_1$$
$$\text{........(54)}$$
$$n_2 = c\sqrt{\mu_2\varepsilon_2} = c/u_2$$

the *refractive indices* of the media.

- Based on these details on oblique incidence, specifically consider <u>two special cases</u>:

  - one with the **E** field perpendicular to the plane of incidence,

  - The other with the **E** field parallel to it.

- Any other polarization may be considered as a <u>linear combination</u> of these two cases.

Electric Polarization

Module – III Part-3

Polarization

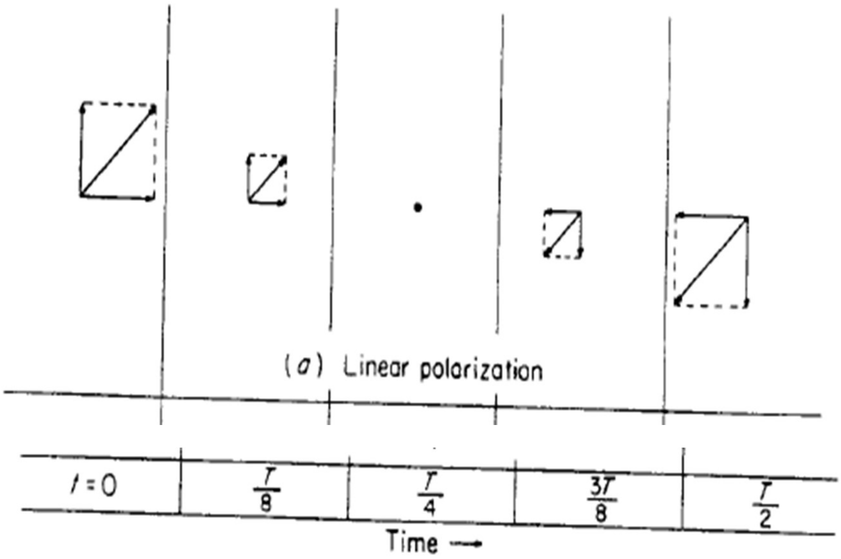
- The polarization of uniform plane wave refers to the time varying behavior of the electric field strength vector at some fined point in space.
- Consider a uniform plane wave travelling in the z direction, with **E** and **H** vectors lying in the x-y plane.
- If $\mathbf{E_y}$=0 and only $\mathbf{E_x}$ is present, the wave is said to be polarized in the x -direction.
- Similarly if $\mathbf{E_x}$ =0 and only $\mathbf{E_y}$ is present the wave is said to be polarized in the y-direction.
- If both $\mathbf{E_x}$ and $\mathbf{E_y}$ are present and are in phase, the resultant electric field has a direction dependent on the relative magnitude of $\mathbf{E_x}$ and $\mathbf{E_y}$.

- The angle which this direction makes with x axis will be constant with time.

$$\therefore \theta = \tan^{-1}\frac{E_y}{E_x} \quad, E = \sqrt{{E_x}^2 + {E_y}^2} \qquad \text{........(1)}$$

Linear polarization

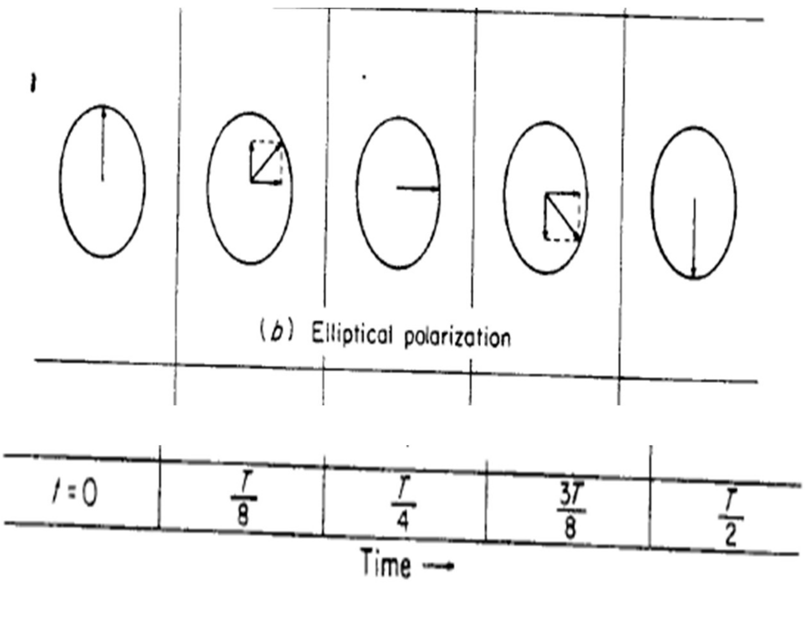
- If both $\mathbf{E}_x$ and $\mathbf{E}_y$ are present and are in phase, the resultant electric field has a direction at an angle of θ.
- If the direction of the resultant vector is constant with time, the wave is said to be linearly polarized.



(a) Linear polarization

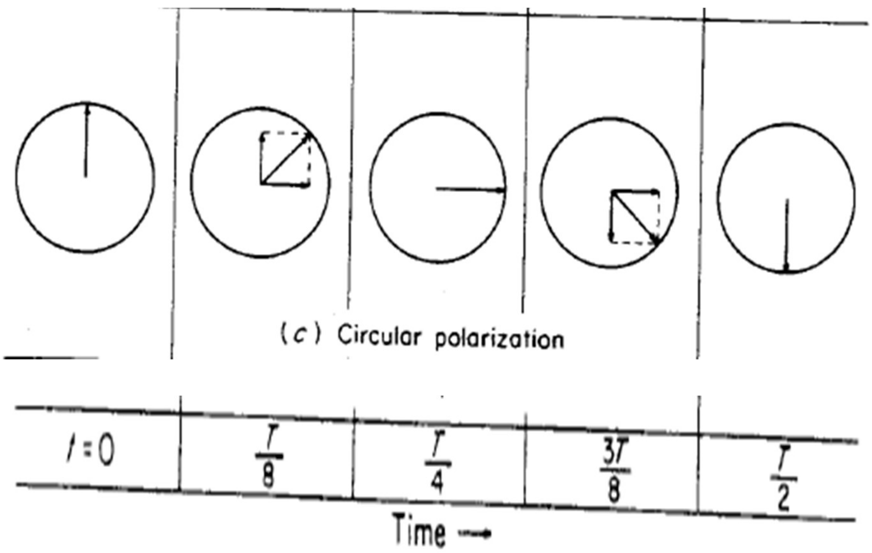
Elliptical Polarization

- If both $\mathbf{E}_x$ and $\mathbf{E}_y$ are present and are not in phase, the traced locus of the tip of the resultant electric field vector will be an Ellipse.
- Such a wave is Elliptically Polarized.



Circular Polarization

- If both $\mathbf{E}_x$ and $\mathbf{E}_y$ are present and are not in phase, have equal magnitudes and a $\pi/2$ phase difference, the locus of the resultant $\mathbf{E}$ is a circle and hence the wave is <u>Circularly Polarized</u>.
- This is a special case of Elliptical Polarization.



The Electric field of a uniform plane wave travelling in the z-direction may be expressed in phasor form as ,

$$\mathbf{E}(z) = \mathbf{E}_0 e^{-j\beta z} \quad \text{........(2)}$$

It may be expressed in time-varying form as ,

$$\tilde{\mathbf{E}}(z, t) = \text{Re}\left\{\mathbf{E}_0 e^{-j\beta z} e^{j\omega t}\right\} \quad \text{........(3)}$$

Since the wave travels in the z-dir, the Electric vector lies in the x-y plane.

$\mathbf{E}_0$ is a complex vector,

$$\mathbf{E}_0 = \mathbf{E}_r + j\mathbf{E}_i \quad \text{........(4)}$$

Here $\mathbf{E}_r$ and $\mathbf{E}_i$ are both real vectors having different directions.

At some point in space, say z=0,

$$\begin{aligned}\tilde{\mathbf{E}}(0, t) &= \text{Re}\left\{(\mathbf{E}_r + j\mathbf{E}_i)e^{j\omega t}\right\} \\ &= \mathbf{E}_r \cos \omega t - \mathbf{E}_i \sin \omega t \quad \text{........(5)}\end{aligned}$$

The time varying electric vector not only changes in <u>magnitude</u>, but also changes its <u>direction</u> as time varies.

Circular Polarization

The *x* and *y* components of Electric field are equal in magnitude.
The y component leads the x component by 90 degrees.
The components have amplitude E_a
The electric field at z=0 is,

$$\mathbf{E}_0 = (\hat{\mathbf{x}} + j\hat{\mathbf{y}})E_a \quad \text{........(6)}$$

The corresponding time-varying field is given by,

$$\tilde{\mathbf{E}}(0, t) = (\hat{\mathbf{x}} \cos \omega t - \hat{\mathbf{y}} \sin \omega t)E_a \quad \text{........(7)}$$

$$\therefore \quad \begin{aligned} \tilde{E}_x &= E_a \cos \omega t \\ \tilde{E}_y &= -E_a \sin \omega t \end{aligned} \quad \text{........(8)}$$

These components satisfy the relation,

$$\tilde{E}_x^2 + \tilde{E}_y^2 = E_a^2 \quad \text{........(9)}$$

This indicates that the endpoint of $\tilde{E}(0, t)$ traces out a circle of radius E_a as time progresses.
The sense or direction of rotation is that of a left-handed screw advancing in the z-dir
So the wave is left circularly polarized.

Similarly right circular polarization is represented by the complex vector,

$$\mathbf{E}_0 = (\hat{\mathbf{x}} - j\hat{\mathbf{y}})E_a \qquad \text{........(10)}$$

A reversal of the sense of rotation is got by a 180-degree phase shift applied either to the x-component or to the y-component of the Electric field.

Elliptical Polarization

- Here x and y components of electric field differ in amplitude.
- Assume again that y component leads the x component by 90 degrees

$$\therefore \qquad \mathbf{E}_0 = \hat{\mathbf{x}}A + j\hat{\mathbf{y}}B \qquad \text{........(11)}$$

- Here A and B are positive real constants.
- The corresponding time-varying field is given by,

$$\tilde{\mathbf{E}}(0, t) = \hat{\mathbf{x}}A\cos\omega t - \hat{\mathbf{y}}B\sin\omega t \qquad \text{........(12)}$$

The components of the time-varying field are,

$$\tilde{E}_x = A \cos \omega t$$
$$\tilde{E}_y = -B \sin \omega t \qquad \text{........(13)}$$

$$\therefore \quad \frac{\tilde{E}_x^2}{A^2} + \frac{\tilde{E}_y^2}{B^2} = 1 \qquad \text{........(14)}$$

Thus the endpoint of the vector traces out an ellipse and the wave is said to be Elliptically Polarized.
The sense of Polarization is again left-handed.

Elliptical Polarization is the most general form of Polarization.

The Polarization is completely specified by the orientation and axial ratio of the polarization ellipse and by the sense in which the endpoint of the electric field vector moves around the ellipse.

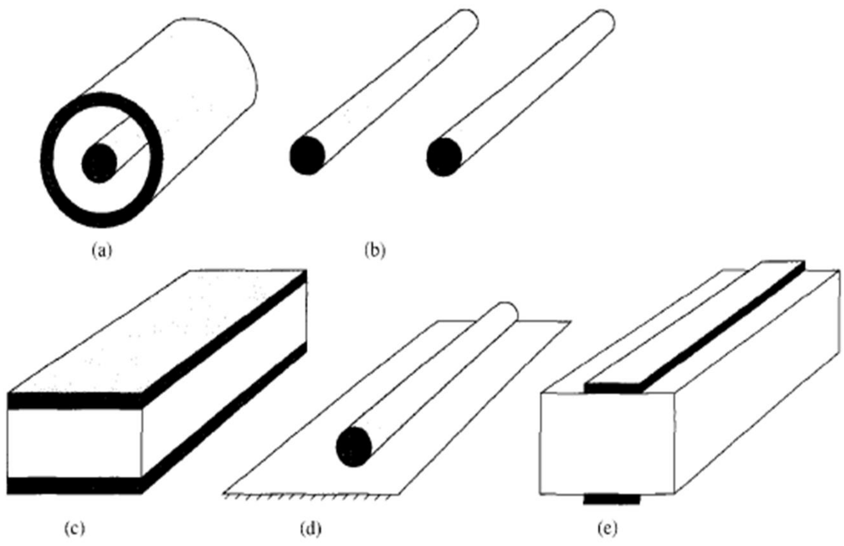
Transmission Lines

EC 303 AET Module-IV

Introduction

- Power or information can be transmitted by guided structures.
- Guided structures serve to guide (or direct) the propagation of energy from the source to the load.
- Typical examples of such structures are Transmission lines and Waveguides.
- Transmission lines are commonly used in Power Distribution (at low frequencies) and in Communications (at high frequencies).
- Various kinds of transmission lines like twisted-pair and coaxial cables (thinnet and thicknet) are used in computer networks such as the Ethernet and internet.

- A transmission line basically consists of two or more parallel conductors used to connect a source to a load.
- The source may be a hydroelectric generator, a transmitter, or an oscillator;
- The load may be a factory, an antenna, or an oscilloscope, respectively.
- Typical transmission lines include coaxial cable, a two-wire line, a parallel-plate or planar line, a wire above the conducting plane, and a microstrip line.

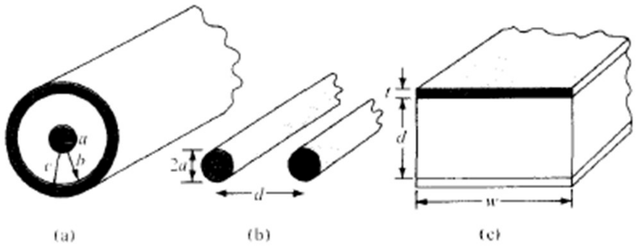


Cross-sectional view of typical transmission lines: **(a)** coaxial line, **(b)** two-wire line, **(c)** planar line, **(d)** wire above conducting plane, **(e)** microstrip line.

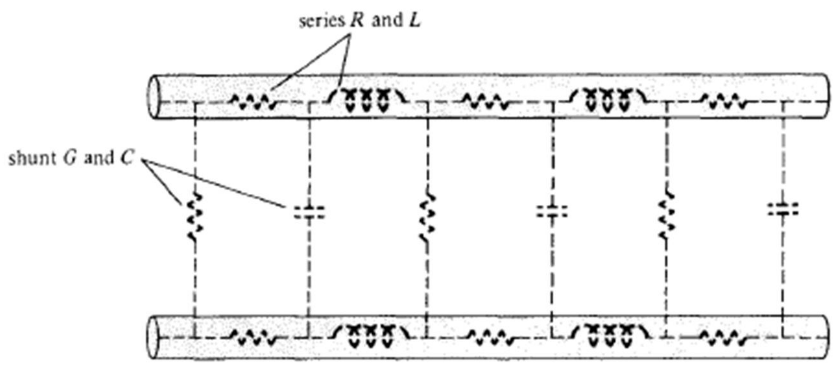
- Note that each of these lines consists of two conductors in parallel.
- Coaxial cables are used in Electrical Laboratories and in connecting TV sets to TV antennas.
- Microstrip lines are particularly important in integrated circuits where metallic strips connecting electronic elements are deposited on dielectric substrates.

TRANSMISSION LINE PARAMETERS

- A transmission line can be analyzed in terms of its line parameters.
- They are:
  - Resistance per unit length *R*
  - Inductance per unit length *L*
  - Conductance per unit length *G*
  - Capacitance per unit length *C*
- The line parameters *R, L, G,* and *C* are not discrete or lumped but distributed.
- The parameters are uniformly distributed along the entire length of the line.



Common transmission lines: **(a)** coaxial line, **(b)** two-wire line, **(c)** planar line.



Distributed parameters of a two-conductor transmission line.

For each line,

$$LC = \mu\varepsilon \qquad \text{and} \qquad \frac{G}{C} = \frac{\sigma}{\varepsilon} \qquad \ldots\ldots\ldots\ldots\ldots(1)$$

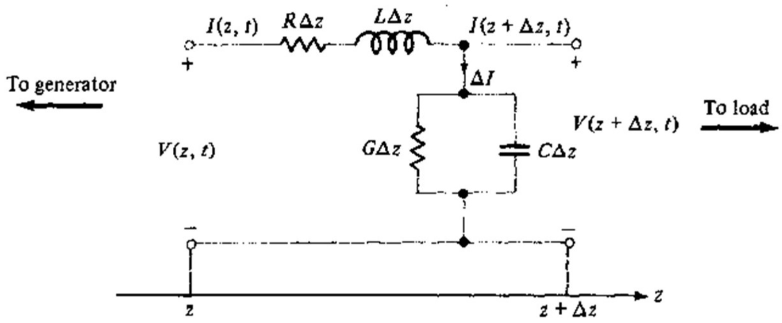
- For each line, the conductors are characterized by σ_c ,ε_c and μ_c and the homogeneous dielectric separating the conductors is characterized by σ, ε and μ.
- $G \neq \frac{1}{R}$; *R* is the ac resistance per unit length of the conductors comprising the line and *G* is the conductance per unit length due to the dielectric medium separating the conductors.
- The value of *L is* the external inductance per unit length; that is, $L = L_{ext}$.
- The effects of internal inductance are negligible at high frequencies at which most communication systems operate.

TRANSMISSION LINE EQUATIONS

- A two-conductor transmission line supports a TEM wave;
- that is, the electric and magnetic fields on the line are transverse to the direction of wave propagation.
- An important property of TEM waves is that the fields **E** and **H** are uniquely related to voltage *V* and current *I*

$$V = -\int \mathbf{E} \cdot d\mathbf{l}, \qquad I = \oint \mathbf{H} \cdot d\mathbf{l} \qquad \text{……………..(2)}$$

Examine an incremental portion of length Δz of a two-conductor transmission line.
To find an equivalent circuit for this line and derive the line equations.



L-type equivalent circuit model of a differential length Δz of a two-conductor transmission line.

- The model is in terms of the line parameters *R, L, G,* and C, and may represent any of the two-conductor lines.
- The model is called the L-type equivalent circuit;
- In the model, assume that the wave propagates along the +z-direction, from generator to the load.
- By applying Kirchhoff's voltage law to the outer loop of the circuit,

$$V(z, t) = R\,\Delta z\, I(z, t) + L\,\Delta z \frac{\partial I(z, t)}{\partial t} + V(z + \Delta z, t)$$

$$-\frac{V(z + \Delta z, t) - V(z, t)}{\Delta z} = R\, I(z, t) + L \frac{\partial I(z, t)}{\partial t} \quad \text{........(3)}$$

Taking the limit as $\Delta z \to 0$ leads to

$$-\frac{\partial V(z,t)}{\partial z} = RI(z,t) + L\frac{\partial I(z,t)}{\partial t} \quad \text{........(4)}$$

Similarly, applying Kirchoff's current law to the main node of the circuit gives

$$\begin{aligned} I(z,t) &= I(z+\Delta z, t) + \Delta I \\ &= I(z+\Delta z, t) + G\,\Delta z\, V(z+\Delta z, t) + C\,\Delta z\frac{\partial V(z+\Delta z, t)}{\partial t} \end{aligned}$$

$$-\frac{I(z+\Delta z, t) - I(z,t)}{\Delta z} = G\,V(z+\Delta z, t) + C\frac{\partial V(z+\Delta z, t)}{\partial t} \quad \text{........(5)}$$

As $\Delta z \to 0$,

$$-\frac{\partial I(z,t)}{\partial z} = G\,V(z,t) + C\frac{\partial V(z,t)}{\partial t} \quad \text{........(6)}$$

assume harmonic time dependence so that

$$V(z,t) = \text{Re}\,[V_s(z)\,e^{j\omega t}] \quad \text{........(7a)}$$

$$I(z,t) = \text{Re}\,[I_s(z)\,e^{j\omega t}] \quad \text{........(7b)}$$

where $V_s(z)$ and $I_s(z)$ are the phasor forms of $V(z,t)$ and $I(z,t)$,

$$-\frac{dV_s}{dz} = (R + j\omega L)\, I_s \qquad \text{........(8)}$$

$$-\frac{dI_s}{dz} = (G + j\omega C)\, V_s \qquad \text{........(9)}$$

Take second derivative of eqn(8) and apply eqn(9)

$$\therefore \quad \frac{d^2V_s}{dz^2} = (R + j\omega L)(G + j\omega C)\, V_s$$

$$\therefore \quad \frac{d^2V_s}{dz^2} - \gamma^2 V_s = 0 \qquad \text{........(10)}$$

where $\gamma = \alpha + j\beta = \sqrt{(R + j\omega L)(G + j\omega C)}$

........(11)

Similarly take second derivative of eqn(9)
and make use of eqn(8),

$$\therefore \quad \frac{d^2I_s}{dz^2} - \gamma^2 I_s = 0 \qquad \text{........(12)}$$

Eqns(10) and (12) are Wave Equations for *V* and *I*

- γ in eq. (11) is the propagation constant (in per meter),
- α is the attenuation constant (in Nepers per meter or Decibels per meter), and
- β is the phase constant (in radians per meter).

- The wavelength λ and wave velocity *u* are, given by,

$$\lambda = \frac{2\pi}{\beta} \qquad \text{........(13)}$$

$$u = \frac{\omega}{\beta} = f\lambda \qquad \text{........(14)}$$

The solutions of the linear homogeneous differential equations (10) and (12) are,

$$V_s(z) = \underbrace{V_o^+ e^{-\gamma z}}_{\longrightarrow +z} + \underbrace{V_o^- e^{\gamma z}}_{-z \longleftarrow} \qquad \text{........(15)}$$

$$I_s(z) = \underbrace{I_o^+ e^{-\gamma z}}_{\longrightarrow +z} + \underbrace{I_o^- e^{\gamma z}}_{-z \longleftarrow} \qquad \text{........(16)}$$

- where V_0^+, V_0^-, I_0^+, and I_0^- are wave amplitudes;
- the + and — signs, respectively, denote wave traveling along *+z-* and -z-directions, as is also indicated by the arrows.

Obtain the instantaneous expression for voltage as,

$$V(z, t) = \text{Re}\,[V_s(z)\, e^{j\omega t}]$$
$$= V_o^+ e^{-\alpha z} \cos(\omega t - \beta z) + V_o^- e^{\alpha z} \cos(\omega t + \beta z)$$

……..(17)

The **characteristic impedance** Z_0 of the line is the ratio of positively traveling voltage wave to current wave at any point on the line.

Z_0 is analogous to γ, the intrinsic impedance of the medium of wave propagation.
By substituting eqs. (15) and (16) into eqs. (8) and (9) and equating coefficients of exponential terms,
terms $e^{\gamma z}$ and $e^{-\gamma z}$,

$$\therefore \quad Z_o = \frac{V_o^+}{I_o^+} = -\frac{V_o^-}{I_o^-} = \frac{R + j\omega L}{\gamma} = \frac{\gamma}{G + j\omega C} \qquad \text{……..(18)}$$

$$\therefore \quad Z_o = \sqrt{\frac{R + j\omega L}{G + j\omega C}} = R_o + jX_o \qquad \text{……..(19)}$$

- where R_0 and X_0 are real and imaginary parts of Z_0.
- R_0 should not be mistaken for R
- while R is in ohms per meter; R_0 is in ohms.

- The propagation constant γ and the characteristic impedance Z_0 are important properties of the line since they both depend on the line parameters *R, L, G,* and *C* and the frequency of operation.

- The reciprocal of Z_0 is the Characteristic Admittance Y_0 that is,

$$\therefore \quad Y_0 = 1/ Z_0$$

The transmission line considered so far is the *lossy* type in that the conductors comprising the line are imperfect ($\sigma_c \neq \infty$) and the dielectric in which the conductors are embedded is lossy ($\sigma \neq 0$).

Consider two special cases of lossless transmission line and distortionless line.

A. Lossless Line (R = 0 = G)

A transmission line is said lo be **lossless** if the conductors of the line are perfect $\sigma_c \approx \infty$) and the dielectric medium separating them is lossless ($\sigma \approx \infty$).

For such a line,

$$\therefore \quad R = 0 = G \qquad \text{........(20)}$$

This is a necessary condition for a line to be lossless.

Thus for such a line, subst. eq. (20) into eqs. (11), (14), and (19),

$$\therefore \quad \alpha = 0, \qquad \gamma = j\beta = j\omega\sqrt{LC} \qquad \text{........(21a)}$$

$$\therefore \quad u = \frac{\omega}{\beta} = \frac{1}{\sqrt{LC}} = f\lambda \qquad \text{........(21b)}$$

$$\therefore \quad X_o = 0, \qquad Z_o = R_o = \sqrt{\frac{L}{C}} \qquad \text{........(21c)}$$

B. Distortionless Line (R/L = G/C)

- A signal normally consists of a band of frequencies;
- Wave amplitudes of different frequency components will be attenuated differently in a lossy line as α is frequency dependent.
- This results in distortion.

A distortionless line is one in which the attenuation constant α is frequency independent while the phase constant β is linearly dependent on frequency.

From the general expression for α and β in eq. (11), a distortionless line results if the line parameters are such that,

$$\frac{R}{L} = \frac{G}{C} \qquad \text{........(22)}$$

Thus, for a distortionless line,

$$\gamma = \sqrt{RG\left(1 + \frac{j\omega L}{R}\right)\left(1 + \frac{j\omega C}{G}\right)}$$
$$= \sqrt{RG}\left(1 + \frac{j\omega C}{G}\right) = \alpha + j\beta$$

$$\therefore \quad \alpha = \sqrt{RG}, \qquad \beta = \omega\sqrt{LC} \qquad \text{........(23a)}$$

This shows that α does not depend on frequency whereas β is a linear function of frequency.

Also,

$$Z_o = \sqrt{\frac{R(1 + j\omega L/R)}{G(1 + j\omega C/G)}} = \sqrt{\frac{R}{G}} = \sqrt{\frac{L}{C}} = R_o + jX_o$$

$$\therefore \quad R_o = \sqrt{\frac{R}{G}} = \sqrt{\frac{L}{C}}, \qquad X_o = 0 \qquad \text{........(23b)}$$

$$\therefore \quad u = \frac{\omega}{\beta} = \frac{1}{\sqrt{LC}} = f\lambda \qquad \text{........(23c)}$$

Points to Ponder

1. The phase velocity is independent of frequency because the phase constant β linearly depends on frequency.
2. There is shape distortion of signals unless α and *u* are independent of frequency.
3. *u* and Z_o remain the same as for lossless lines.

Points to Ponder (Cont…)

4. A lossless line is also a distortionless line, but a distortionless line is not necessarily lossless.
5. Although lossless lines are desirable in power transmission, telephone lines are required to be distortionless.

Transmission Line Characteristics

Case	Propagation Constant $\gamma = \alpha + j\beta$	Characteristic Impedance $Z_o = R_o + jX_o$
General	$\sqrt{(R + j\omega L)(G + j\omega C)}$	$\sqrt{\dfrac{R + j\omega L}{G + j\omega C}}$
Lossless	$0 + j\omega\sqrt{LC}$	$\sqrt{\dfrac{L}{C}} + j0$
Distortionless	$\sqrt{RG} + j\omega\sqrt{LC}$	$\sqrt{\dfrac{L}{C}} + j0$

Problem

An air line has characteristic impedance of 70 Ω and phase constant of 3 rad/m at 100 MHz. Calculate the inductance per meter and the capacitance per meter of the line.

Sol:

An air line can be regarded as a lossless line since $\sigma \approx 0$.

$$\therefore \quad R = 0 = G \qquad \text{and} \qquad \alpha = 0$$

$$\therefore \quad Z_o = R_o = \sqrt{\frac{L}{C}} \qquad \text{........(P-1)}$$

$$\therefore \quad \beta = \omega\sqrt{LC} \qquad \text{........(P-2)}$$

Dividing eq. (P-1) by eq. (P-2) yields,

$$\therefore \quad \frac{R_o}{\beta} = \frac{1}{\omega C}$$

$$\therefore \; C = \frac{\beta}{\omega R_o} = \frac{3}{2\pi \times 100 \times 10^6 (70)} = \underline{\underline{68.2 \text{ pF/m}}}$$

From eq. (1),

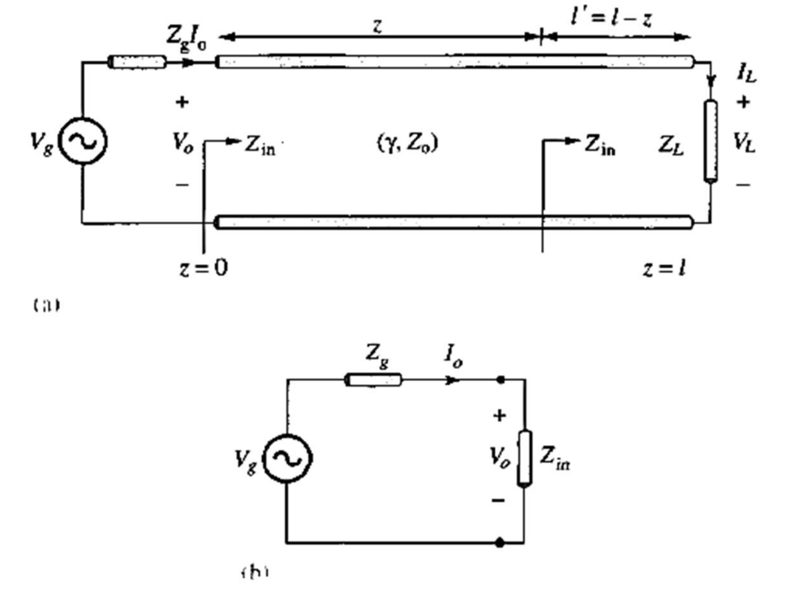
$$\therefore \; L = R_o^2 C = (70)^2 (68.2 \times 10^{-12}) = \underline{\underline{334.2 \text{ nH/m}}}$$

INPUT IMPEDANCE, SWR, AND POWER

Consider a transmission line of length ℓ, characterized by γ and Z_0 connected to a load Z_L.

Looking into the line, the generator sees the line with the load as an input impedance Z_{in}.

Need to determine the following:
- input impedance,
- the standing wave ratio (SWR), and
- the power flow on the line.



(a) Input impedance due to a line terminated by a load; **(b)** equivalent circuit for finding V_o and I_o in terms of Z_{in} at the input.

Let the transmission line extend from z = 0 at the generator to z = ℓ at the load.
Need to find the voltage and current waves using eqs. (15) and (16), i.e.,

$$V_s(z) = V_o^+ e^{-\gamma z} + V_o^- e^{\gamma z} \qquad \text{........(24)}$$

$$I_s(z) = \frac{V_o^+}{Z_o} e^{-\gamma z} - \frac{V_o^-}{Z_o} e^{\gamma z} \qquad \text{........(25)}$$

where, $Z_o = \dfrac{V_o^+}{I_o^+}$;

Let, $V_o = V(z = 0), \qquad I_o = I(z = 0)$ ....(26)

$$\therefore \quad V_o^+ = \frac{1}{2}(V_o + Z_o I_o) \qquad \text{........(27a)}$$

$$\therefore \quad V_o^- = \frac{1}{2}(V_o - Z_o I_o) \qquad \text{........(27b)}$$

If the input impedance at the input terminals is Z_{in} , the input voltage V_0 and the input current I_0 are,

$$V_o = \frac{Z_{in}}{Z_{in} + Z_g} V_g, \qquad I_o = \frac{V_g}{Z_{in} + Z_g} \qquad \text{........(28)}$$

Given the condition for the lord,

$$V_L = V(z = \ell), \qquad I_L = I(z = \ell) \qquad \text{........(29)}$$

Substituting these into eqs. (24) and (25) gives,

$$V_o^+ = \frac{1}{2}(V_L + Z_o I_L)e^{\gamma\ell} \qquad \text{........(30a)}$$

$$V_o^- = \frac{1}{2}(V_L - Z_o I_L)e^{-\gamma\ell} \qquad \text{........(30b)}$$

Determine the input impedance Z_{in} at any point on line. At the generator, eqs. (24) and (25) yield,

$$Z_{in} = \frac{V_s(z)}{I_s(z)} = \frac{Z_o(V_o^+ + V_o^-)}{V_o^+ - V_o^-} \qquad \text{........(31)}$$

Substitute eq. (30) into (31) and utilizing the fact that

$$\frac{e^{\gamma\ell} + e^{-\gamma\ell}}{2} = \cosh \gamma\ell, \qquad \frac{e^{\gamma\ell} - e^{-\gamma\ell}}{2} = \sinh \gamma\ell \qquad \text{........(32a)}$$

$$\tanh \gamma\ell = \frac{\sinh \gamma\ell}{\cosh \gamma\ell} = \frac{e^{\gamma\ell} - e^{-\gamma\ell}}{e^{\gamma\ell} + e^{-\gamma\ell}} \qquad \text{........(32b)}$$

$$\therefore \quad Z_{in} = Z_o \left[\frac{Z_L + Z_o \tanh \gamma\ell}{Z_o + Z_L \tanh \gamma\ell} \right] \text{ (lossy)} \qquad \text{........(33)}$$

Although eq. (33) has been derived for the input impedance Z_{in} at the generation end, it is a general expression for finding Z_{in} at any point on the line.

For a lossless line, $\gamma = j\beta$, $\tanh j\beta\ell = j \tan \beta\ell$, $Z_o = R_o$

$$\therefore \; Z_{in} = Z_o \left[\frac{Z_L + jZ_o \tan \beta\ell}{Z_o + jZ_L \tan \beta\ell} \right] \quad \text{(lossless)} \qquad \ldots\ldots..(34)$$

the input impedance varies periodically with distance ℓ from the load.

The quantity $\beta\ell$ in eq. (34) is called the Electrical length of the line and can be expressed in degrees or radians.

- Define Γ_L as the *voltage reflection coefficient* (at the load).
- It is the ratio of the voltage reflection wave to the incident wave at the load, i.e.,

$$\therefore \; \Gamma_L = \frac{V_o^- e^{\gamma\ell}}{V_o^+ e^{-\gamma\ell}} \qquad \ldots\ldots..(35)$$

- Substitute V_0^- and V_0^+ from eq. (30) into eq. (35) and incorporating $V_L = Z_L.I_L$ gives,

$$\therefore \; \Gamma_L = \frac{Z_L - Z_o}{Z_L + Z_o} \qquad \ldots\ldots..(36)$$

The **voltage reflection coefficient** at any point on the line is the ratio of the magnitude of reflected voltage wave to that of incident wave.

$$\therefore \quad \Gamma(z) = \frac{V_o^- e^{\gamma z}}{V_o^+ e^{-\gamma z}} = \frac{V_o^-}{V_o^+} e^{2\gamma z}$$

But $z = \ell - \ell'$

Substitute and combining with eq. (35),

$$\therefore \quad \Gamma(z) = \frac{V_o^-}{V_o^+} e^{2\gamma\ell} e^{-2\gamma\ell'} = \Gamma_L e^{-2\gamma\ell'} \qquad \text{........(37)}$$

The **current reflection coefficient** at any point on the line is negative of the voltage reflection coefficient at that point.

Thus, the current reflection coefficient at load is,

$$I_o^- e^{\gamma\ell} / I_o^+ e^{-\gamma\ell} = -\Gamma_L.$$

Define the *Standing Wave Ratio* (SWR), *s* as,

$$\therefore \quad s = \frac{V_{max}}{V_{min}} = \frac{I_{max}}{I_{min}} = \frac{1 + |\Gamma_L|}{1 - |\Gamma_L|} \qquad \text{........(38)}$$

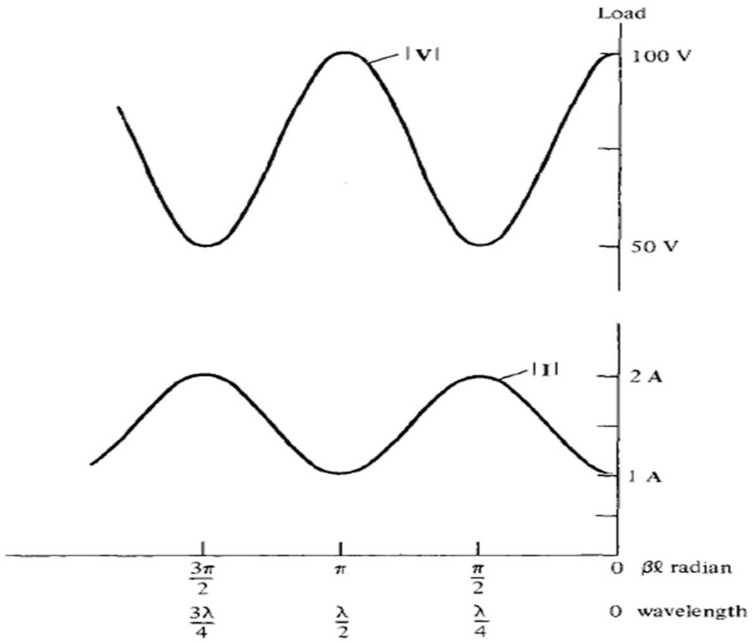
The input impedance Z_{in} has maxima and minima that occur, respectively, at the maxima and minima of the voltage and current standing wave.

$$\therefore \quad |Z_{in}|_{max} = \frac{V_{max}}{I_{min}} = sZ_o \quad \text{........(39a)}$$

$$\therefore \quad |Z_{in}|_{min} = \frac{V_{min}}{I_{max}} = \frac{Z_o}{s} \quad \text{........(39b)}$$

- Consider a lossless line with characteristic impedance of $Z_o = 50\ \Omega$.
- Assume that the line is terminated in a pure resistive load $Z_L = 100\ \Omega$
- Let Voltage at the load is 100 V (rms).

- The conditions on the line are displayed.
- Note from the figure that conditions on the line repeat themselves every half wavelength.



Voltage and current wave patterns on a lossless line terminated by a resistive load.

Power on Transmission line

A transmission is used in transferring power from the source to the load.
The average input power at a distance ℓ from the load is,

$$\therefore \quad P_{\text{ave}} = \frac{1}{2} \text{Re}\,[V_s(\ell) I_s^*(\ell)]$$

the factor $\frac{1}{2}$ is needed as the peak values are used instead of the rms values.

Assume a lossless line, substitute eqs. (24) and (25) to obtain,

$$\therefore \; P_{ave} = \frac{1}{2}\text{Re}\left[V_o^+(e^{j\beta\ell} + \Gamma e^{-j\beta\ell}) \frac{V_o^{+*}}{Z_o}(e^{-j\beta\ell} - \Gamma^* e^{j\beta\ell}) \right]$$

$$= \frac{1}{2}\text{Re}\left[\frac{|V_o^+|^2}{Z_o}(1 - |\Gamma|^2 + \Gamma e^{-2j\beta\ell} - \Gamma^* e^{2j\beta\ell}) \right]$$

Since the last two terms are purely imaginary,

$$\therefore \; P_{ave} = \frac{|V_o^+|^2}{2Z_o}(1 - |\Gamma|^2) \qquad \text{........(40)}$$

- The first term is the incident power P_i while the second term is the reflected power P_r.

$$\therefore \; P_t = P_i - P_r$$

- where P_t is the input or transmitted power.
- The power is constant and does not depend on ℓ since it is a lossless line.
- Also, notice that maximum power is delivered to the load when $\Gamma = 0$, as expected.
- Consider special cases when the line is connected to load $Z_L = 0$, $Z_L = \infty$ and $Z_L = Z_o$.
- They can easily be derived from general case.

A. Shorted Line ($Z_\ell = 0$)
For this case, eq. (34) becomes,

$$\therefore Z_{sc} = Z_{in}\Big|_{Z_L=0} = jZ_o \tan \beta\ell \qquad \text{........(41a)}$$

$$\therefore \Gamma_L = -1, \qquad s = \infty \qquad \text{........(41b)}$$

Z_{in} is a pure reactance, which could be capacitive or inductive depending on the value of ℓ.

B. Open-Circuited Line($Z_\ell = \infty$)
For this case, substitute in eq. (34),

$$\therefore Z_{oc} = \lim_{Z_L \to \infty} Z_{in} = \frac{Z_o}{j \tan \beta\ell} = -jZ_o \cot \beta\ell \qquad \text{.....(42a)}$$

$$\therefore \Gamma_L = -1, \qquad s = \infty \qquad \text{........(42b)}$$

Check eqs. (41a) and (42a) ,

$$\therefore Z_{sc}Z_{oc} = Z_o^2 \qquad \text{........(43)}$$

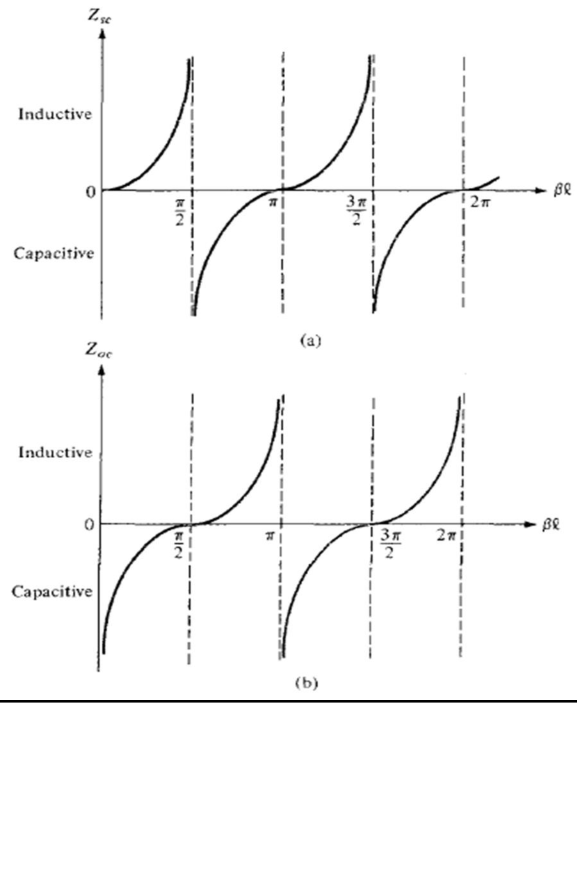
C. Matched Line ($Z_L = Z_0$)

The most desired case from practical view point.
For this case, substitute condition in eq. (34),

$$\therefore \quad Z_{in} = Z_o \quad \text{........(44a)}$$

$$\therefore \quad \Gamma_L = 0, \qquad s = 1 \quad \text{........(44b)}$$

So the whole wave is transmitted and there is no reflection.
The incident power is fully absorbed by load.
Thus maximum power transfer is possible when a transmission line is matched to the load.



Input impedance of a lossless line: **(a)** when shorted, **(b)** when open.

From eqn(34), the input impedance of a lossy line can be written as ,

$$\therefore \quad Z_{\text{in}} = Z_0 \left[\frac{Z_L \cos(\beta l) + jZ_0 \sin(\beta l)}{Z_0 \cos(\beta l) + jZ_L \sin(\beta l)} \right] \qquad \text{........(45)}$$

One special case is that in which the line length is a half-wavelength, or an integer multiple thereof. In that case,

$$\therefore \quad \beta l = \frac{2\pi}{\lambda}\frac{m\lambda}{2} = m\pi \quad (m = 0, 1, 2, \ldots) \qquad \text{........(46)}$$

Using this result in (45),

$$\therefore \quad Z_{\text{in}}(l = m\lambda/2) = Z_L \qquad \text{........(47)}$$

- For a half-wave line, the equivalent circuit can be constructed simply by removing the line completely and placing the load impedance at the input.
- This applies only if the line length is indeed an integer multiple of a half wavelength.
- Once the frequency begins to vary, the condition is no longer satisfied, and (45) must be used in its general form to find Z_{in}.

Another important special case is that in which the line length is an odd multiple of a quarter wavelength:

........(48)

$$\therefore\ \beta l = \frac{2\pi}{\lambda}(2m+1)\frac{\lambda}{4} = (2m+1)\frac{\pi}{2} \quad (m = 0, 1, 2, \ldots)$$

Use this result in (45),

$$\therefore\ Z_{\text{in}}(l = \lambda/4) = \frac{Z_0^2}{Z_L}$$ ....(49)

- An application of (49) is to the problem of joining two lines having different characteristic impedances.
- Suppose the impedances are, (from left to right) Z_{01} and Z_{03}
- At the joint, insert an additional line whose characteristic impedance is Z_{02} and whose length is $\lambda/4$.

So a sequence of joined lines whose impedances progress as Z_{01}, Z_{02}, and Z_{03}, in that order.

A voltage wave is now incident from line 1 onto the joint between Z_{01} and Z_{02}.

Now the effective load at far end of line 2 is Z_{03}. The input impedance to line 2 at any frequency becomes,

$$\therefore \quad Z_{\text{in}} = Z_{02}\frac{Z_{03}\cos\beta_2 l + jZ_{02}\sin\beta_2 l}{Z_{02}\cos\beta_2 l + jZ_{03}\sin\beta_2 l} \qquad \text{........(50)}$$

Then, since the length of line 2 is $\lambda/4$,

$$\therefore \quad Z_{\text{in}}(\text{line } 2) = \frac{Z_{02}^2}{Z_{03}} \qquad \text{........(51)}$$

Reflections at the Z_{01}–Z_{02} interface will not occur if $Z_{\text{in}} = Z_{01}$.

Therefore, we can match the junction (allowing complete transmission through the three-line sequence) if Z_{02} is chosen so that,

$$Z_{02} = \sqrt{Z_{01} Z_{03}} \quad \text{........(52)}$$

This technique is called <u>Quarter-wave matching.</u>

It is limited to the narrow band of frequencies such that,

$$\ell = (2m + 1)\lambda/4 \quad \text{........(53)}$$

TRANSMISSION LINES AND SMITH CHART

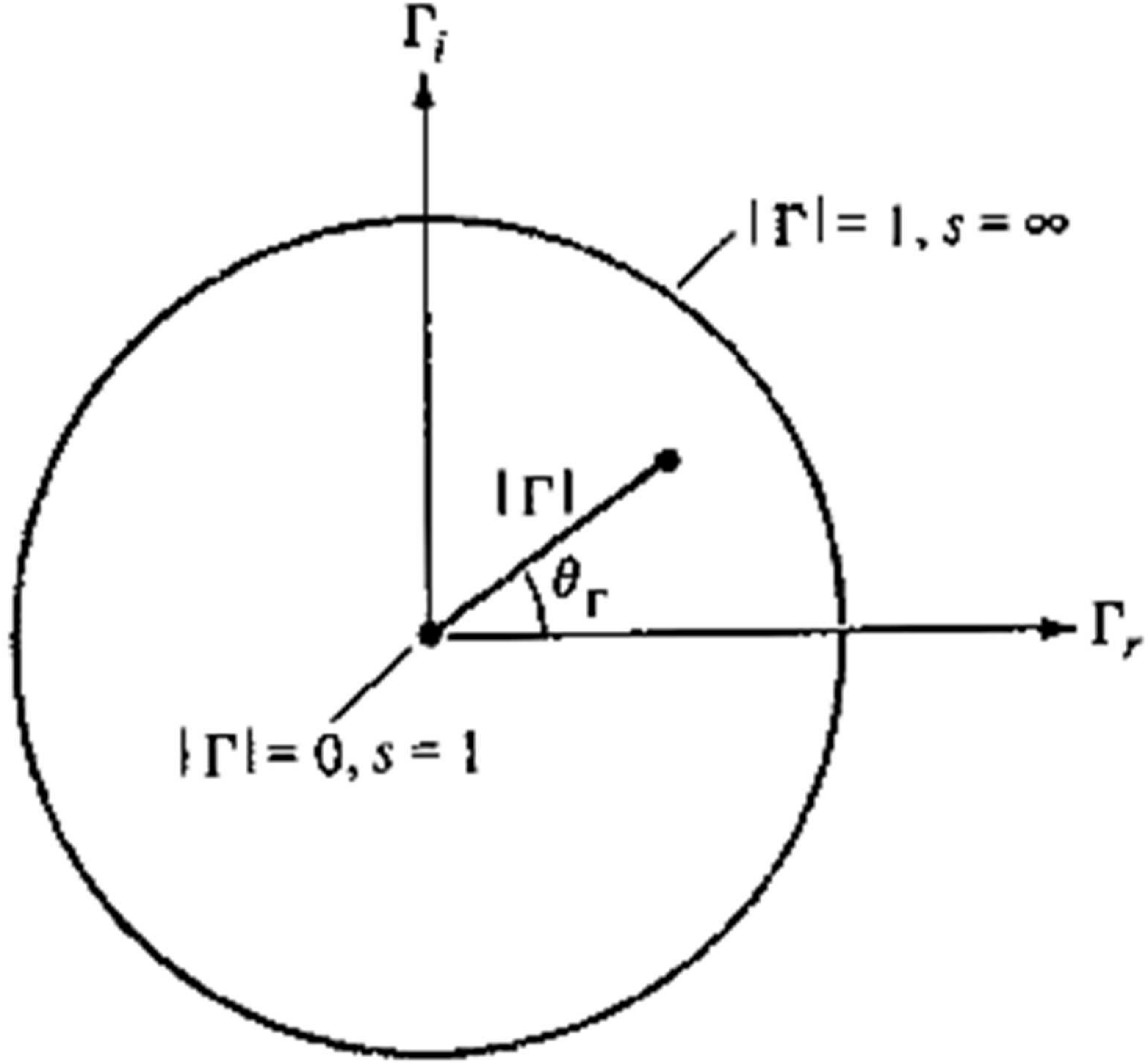
EC303 MODULE-V

THE SMITH CHART

- The Smith chart is the most commonly used graphical technique.
- It is basically a graphical indication of the impedance of a transmission line as one moves along the line.
- The Smith chart is constructed within a circle of unit radius ($|\Gamma| \leq 1$)
- The creation of the chart is based on relation,

$$\Gamma = \frac{Z_L - Z_o}{Z_L + Z_o} \quad \text{........(1)}$$

$$\Gamma = |\Gamma| \underline{/\theta_\Gamma} = \Gamma_r + j\Gamma_i \quad \text{........(2)}$$



Unit circle on which the **Smith chart** is constructed.

- Use a normalized chart in which all impedances are normalized with respect to the characteristic impedance Z_o of the particular line under consideration.
- For the load impedance Z_L for example, the *normalized impedance* z_L is given by,

$$z_L = \frac{Z_L}{Z_o} = r + jx \quad \text{........(3)}$$

$$\therefore \quad \Gamma = \Gamma_r + j\Gamma_i = \frac{z_L - 1}{z_L + 1} \quad \text{........(4a)}$$

$$\therefore \quad z_L = r + jx = \frac{(1 + \Gamma_r) + j\Gamma_i}{(1 - \Gamma_r) - j\Gamma_i} \quad \text{........(4b)}$$

Normalizing and equating components,

$$r = \frac{1 - \Gamma_r^2 - \Gamma_i^2}{(1 - \Gamma_r)^2 + \Gamma_i^2} \quad \text{........(5a)}$$

$$x = \frac{2\,\Gamma_i}{(1 - \Gamma_r)^2 + \Gamma_i^2} \quad \text{........(5b)}$$

$$\left[\Gamma_r - \frac{r}{1 + r}\right]^2 + \Gamma_i^2 = \left[\frac{1}{1 + r}\right]^2 \quad \text{........(6)}$$

$$[\Gamma_r - 1]^2 + \left[\Gamma_i - \frac{1}{x}\right]^2 = \left[\frac{1}{x}\right]^2 \quad \text{........(7)}$$

Each of eqs. (6) and (7) is similar to,

$$(x - h)^2 + (y - k)^2 = a^2 \qquad \text{........(8)}$$

which is the general equation of a circle of radius *a,* centered at *(h, k).*

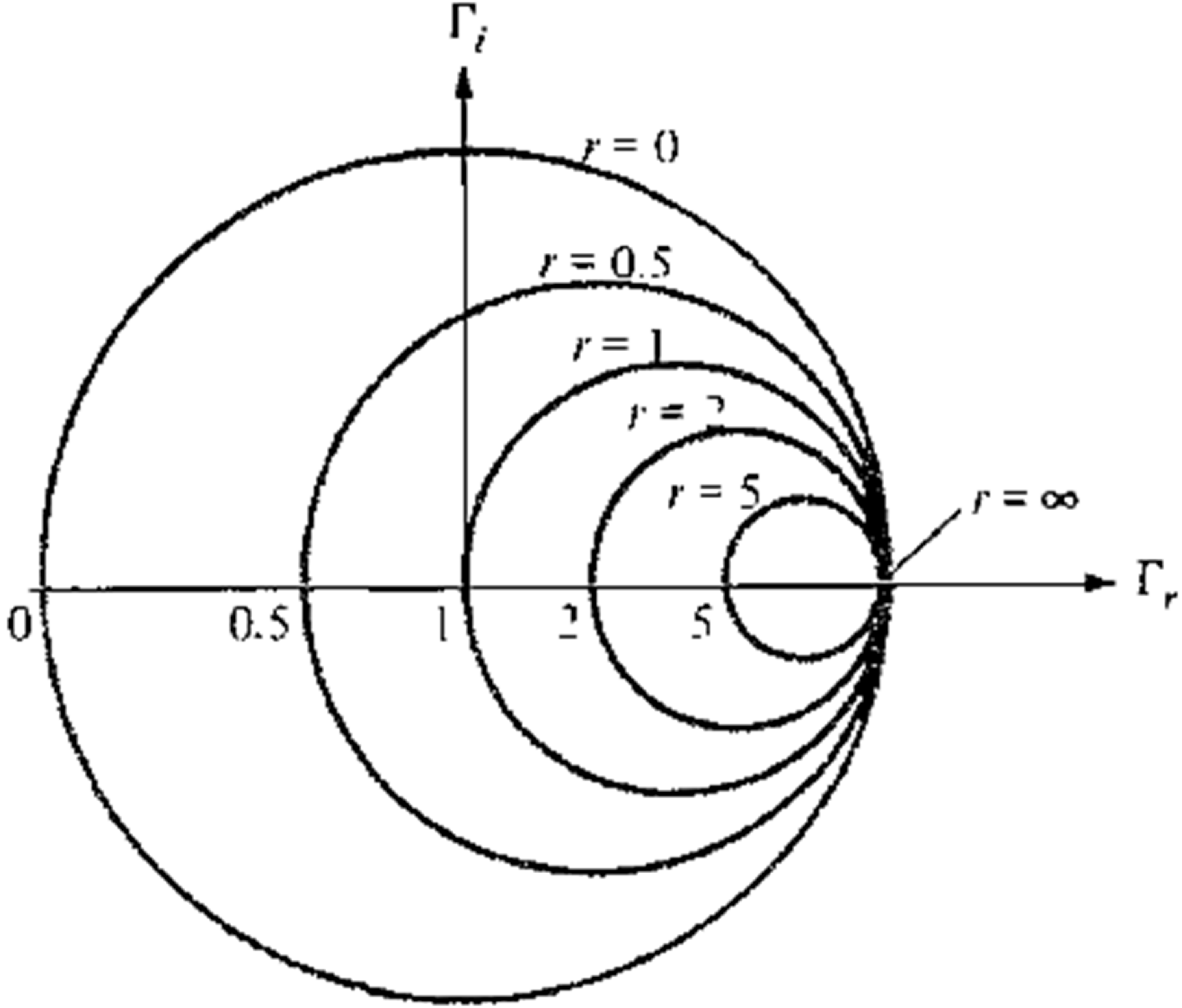
Thus eq. (6) is an r-circle *(resistance circle)* with,

$$\text{center at } (\Gamma_r, \Gamma_i) = \left(\frac{r}{1+r}, 0\right) \qquad \text{........(9a)}$$

$$\text{radius} = \frac{1}{1+r} \qquad \text{........(9b)}$$

Radii and Centers of r-Circles for Typical Values of r

Normalized Resistance (r)	Radius $\left(\frac{1}{1+r}\right)$	Center $\left(\frac{r}{1+r}, 0\right)$
0	1	(0, 0)
1/2	2/3	(1/3, 0)
1	1/2	(1/2, 0)
2	1/3	(2/3, 0)
5	1/6	(5/6, 0)
∞	0	(1, 0)



Typical r-circles for r = 0,0.5, 1,2, 5, ∞

Similarly, eq. (7) is an *x-circle (reactance circle)* with,

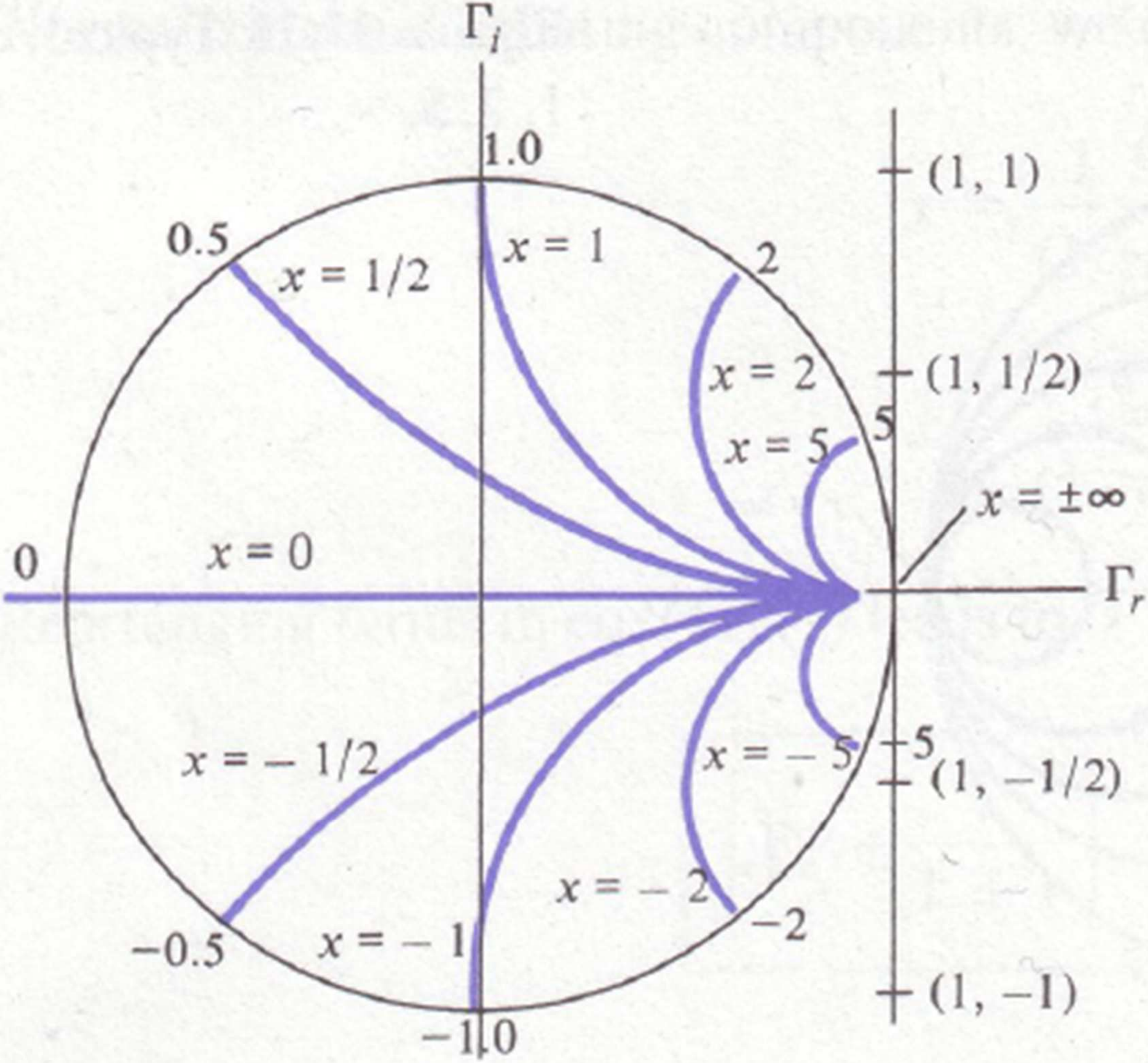
$$\text{center at } (\Gamma_r, \Gamma_i) = \left(1, \frac{1}{x}\right) \qquad \text{........(10a)}$$

$$\text{radius} = \frac{1}{x} \qquad \text{........(10b)}$$

Radii and Centers of x-Circles for Typical Value of *x*

Normalized Reactance (x)	Radius $\left(\frac{1}{x}\right)$	Center $\left(1, \frac{1}{x}\right)$
0	∞	$(1, \infty)$
$\pm 1/2$	2	$(1, \pm 2)$
± 1	1	$(1, \pm 1)$
± 2	1/2	$(1, \pm 1/2)$
± 5	1/5	$(1, \pm 1/5)$
$\pm \infty$	0	$(1, 0)$

Typical x-circles for *x* = 0, ± 1/2,±1, ±2, ±5, ±∞

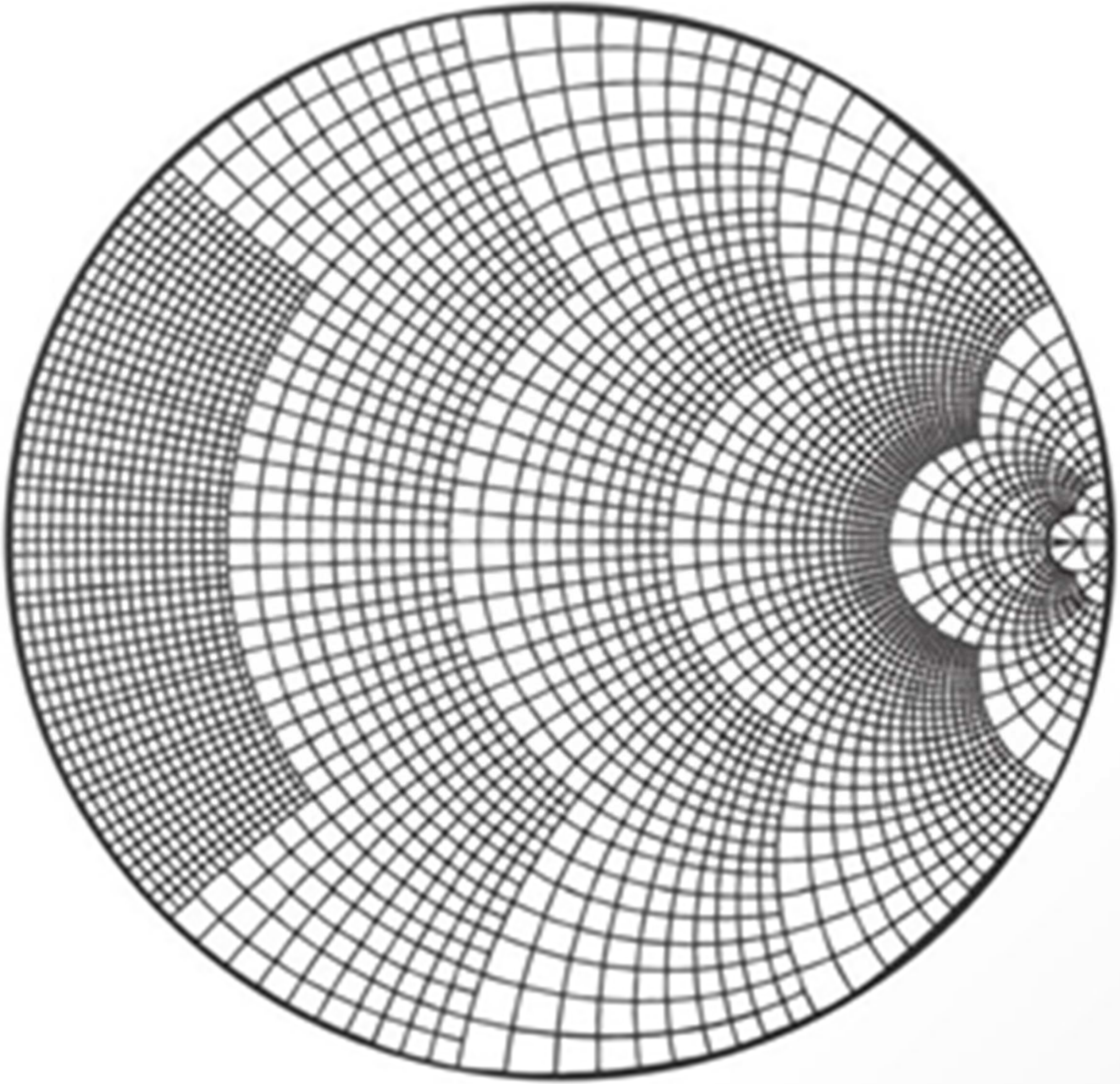


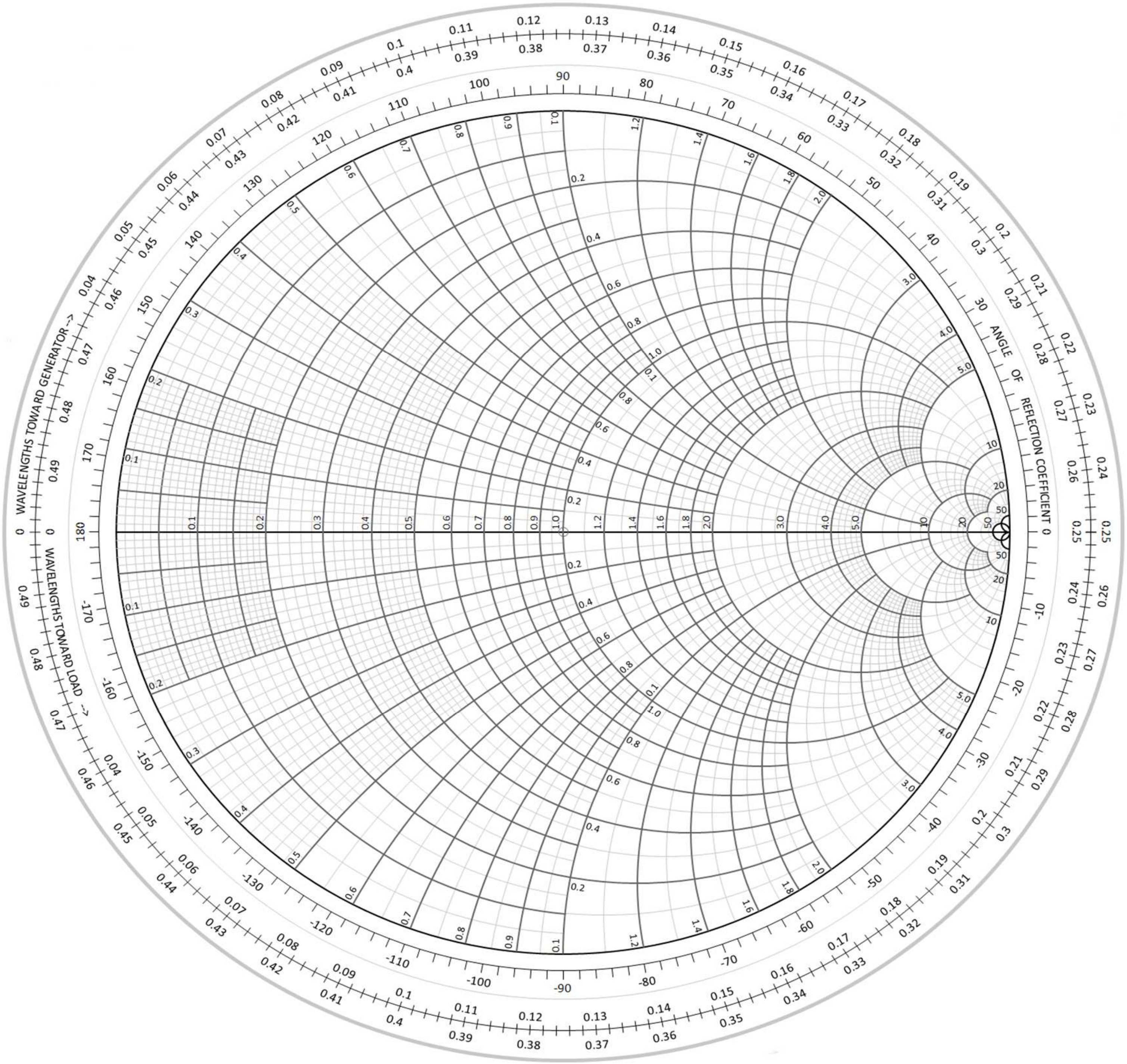
While r is always positive, x can be positive (for inductive impedance) or negative (for capacitive impedance).

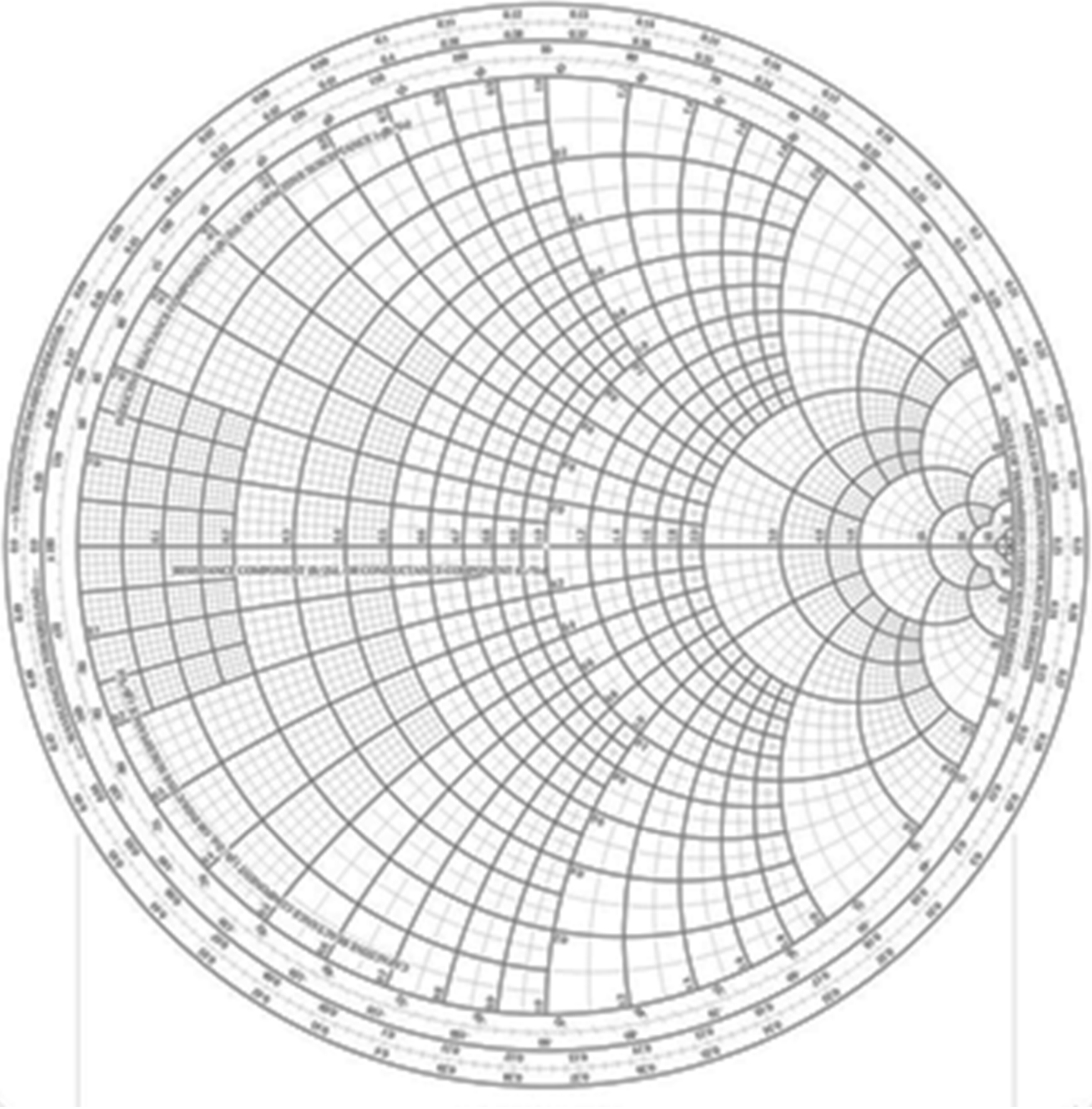
Superpose the r-circles and x-circles, to obtain the **Smith chart**.

The *s-circles* or *constant standing-wave-ratio circles* which are centered at the origin with s varying from 1 to ∞.

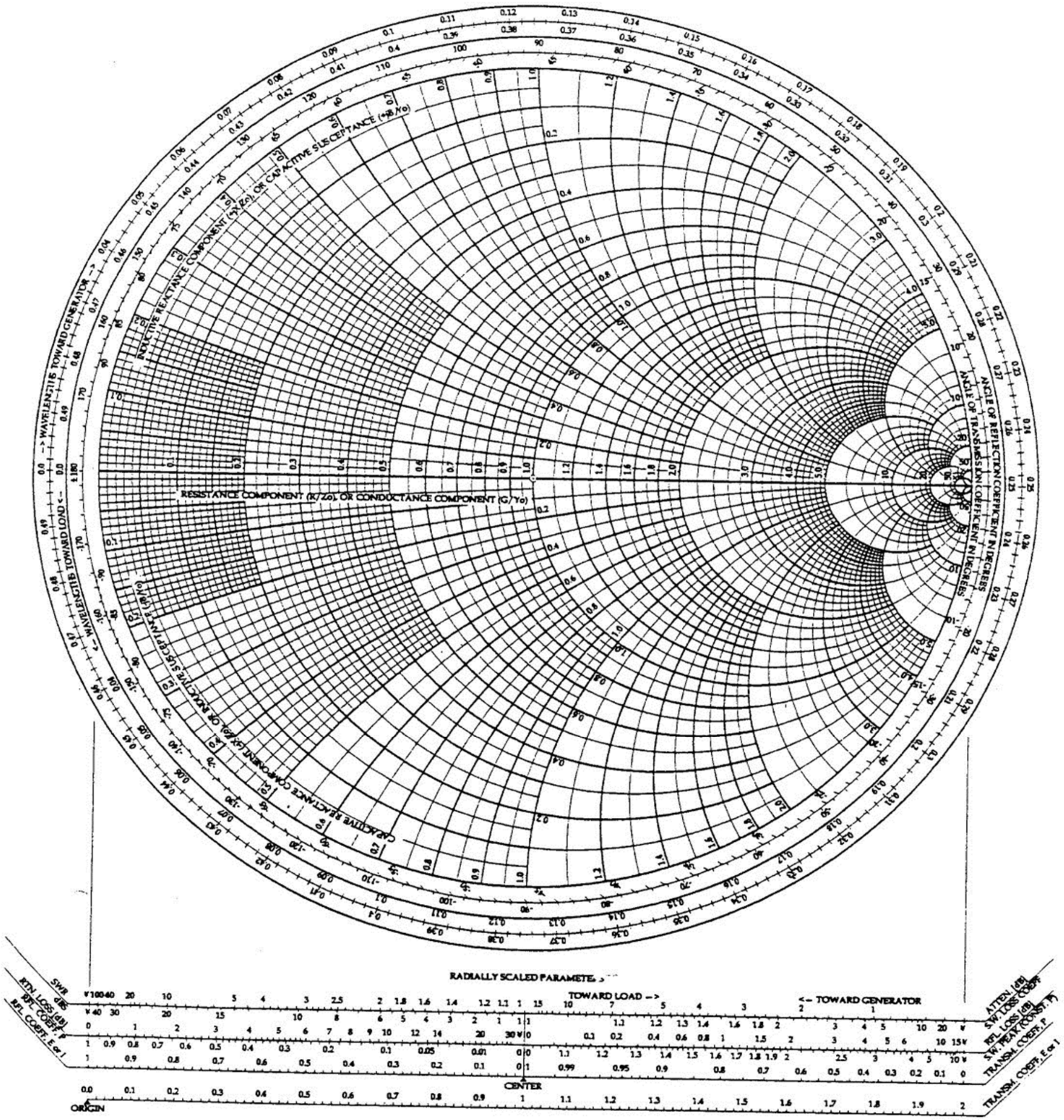
The value of the standing wave ratio s is determined by locating where an s-circle crosses the Γ_r axis. Typical examples of s-circles for s = 1,2, 3, and ∞ are shown.



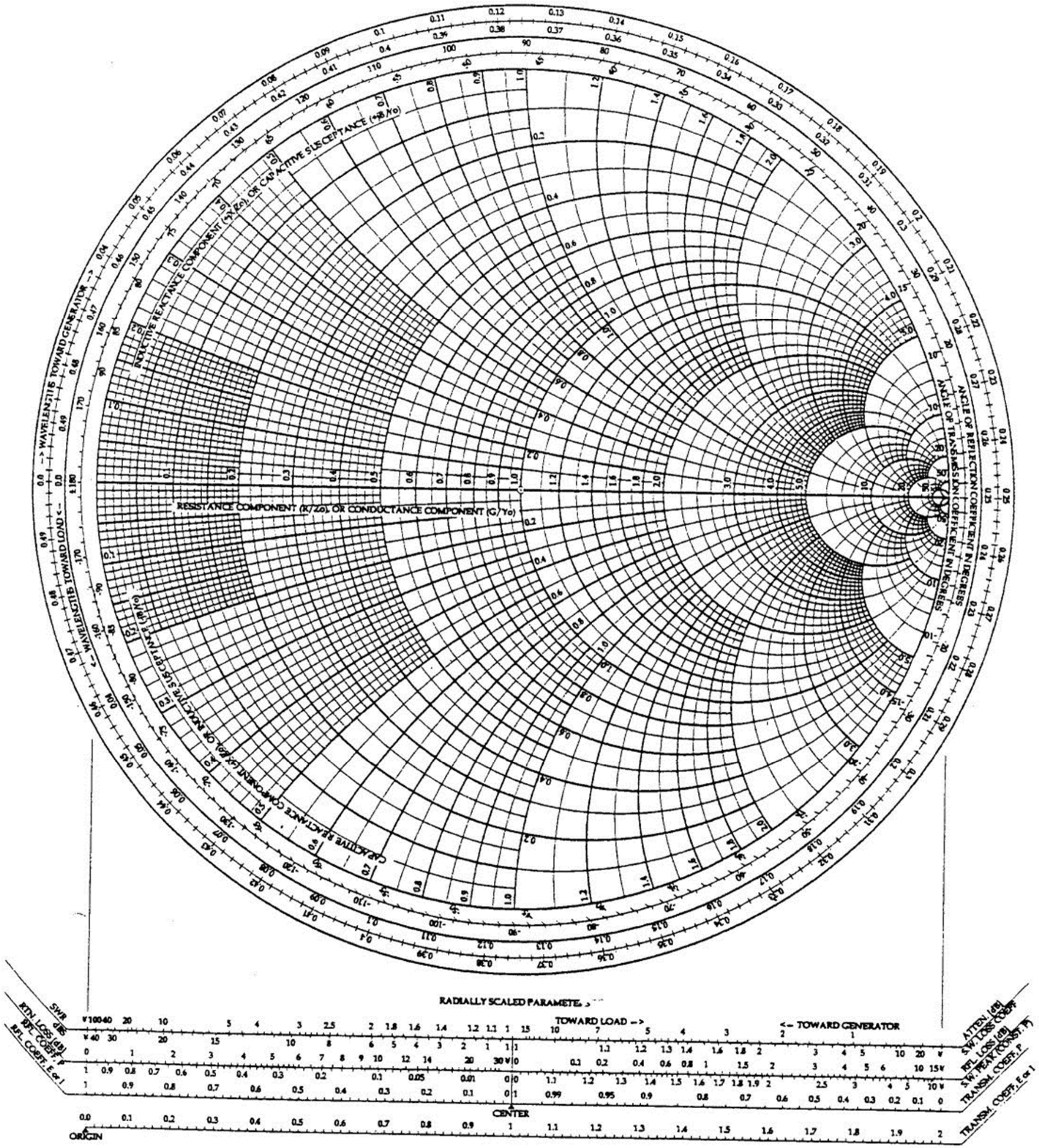

WAVELENGTHS TOWARD GENERATOR -->
WAVELENGTHS TOWARD LOAD -->
ANGLE OF REFLECTION COEFFICIENT

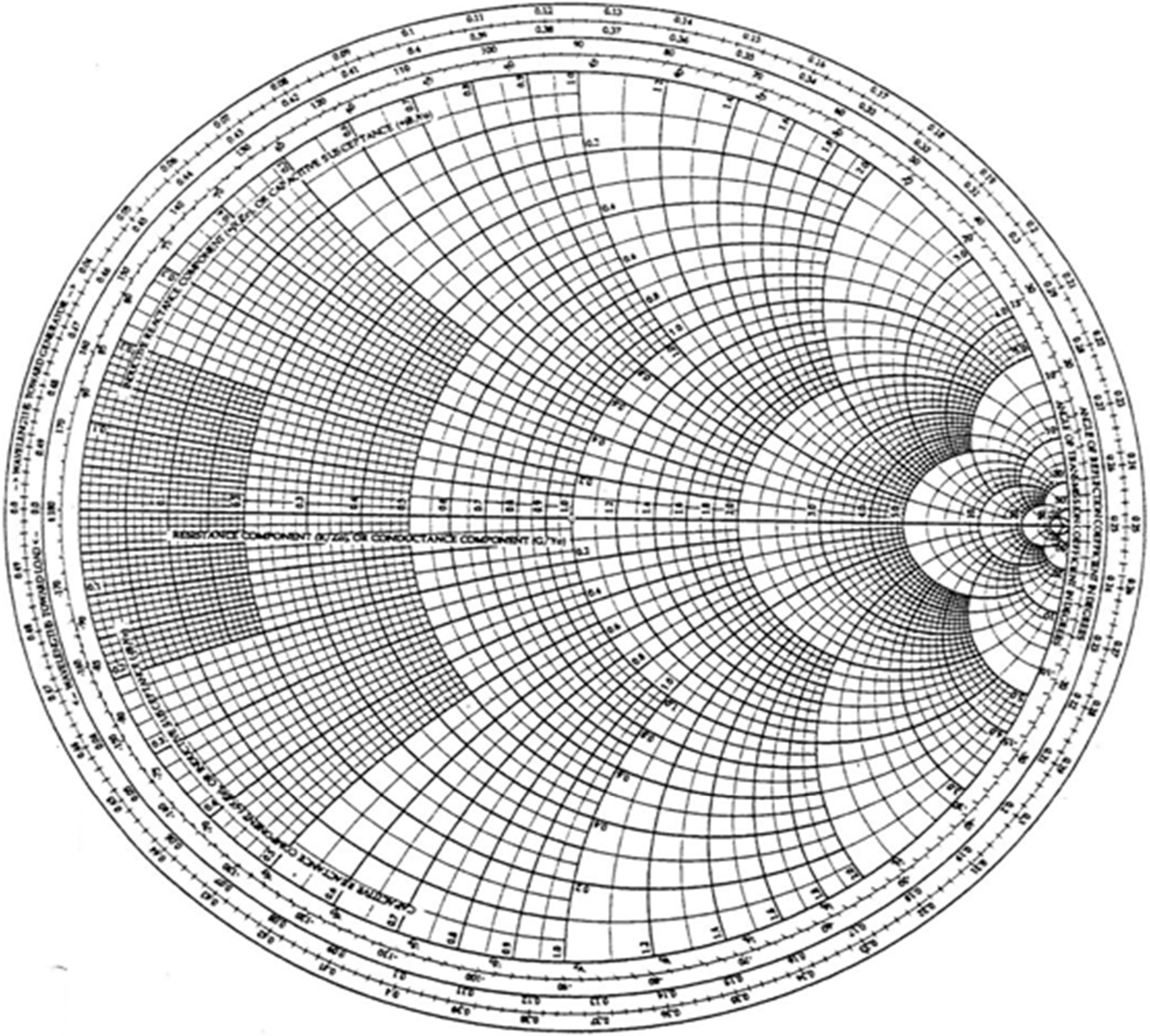


The Smith Chart



The Smith Chart





RESISTANCE COMPONENT (R/Zo), OR CONDUCTANCE COMPONENT (G/Yo)

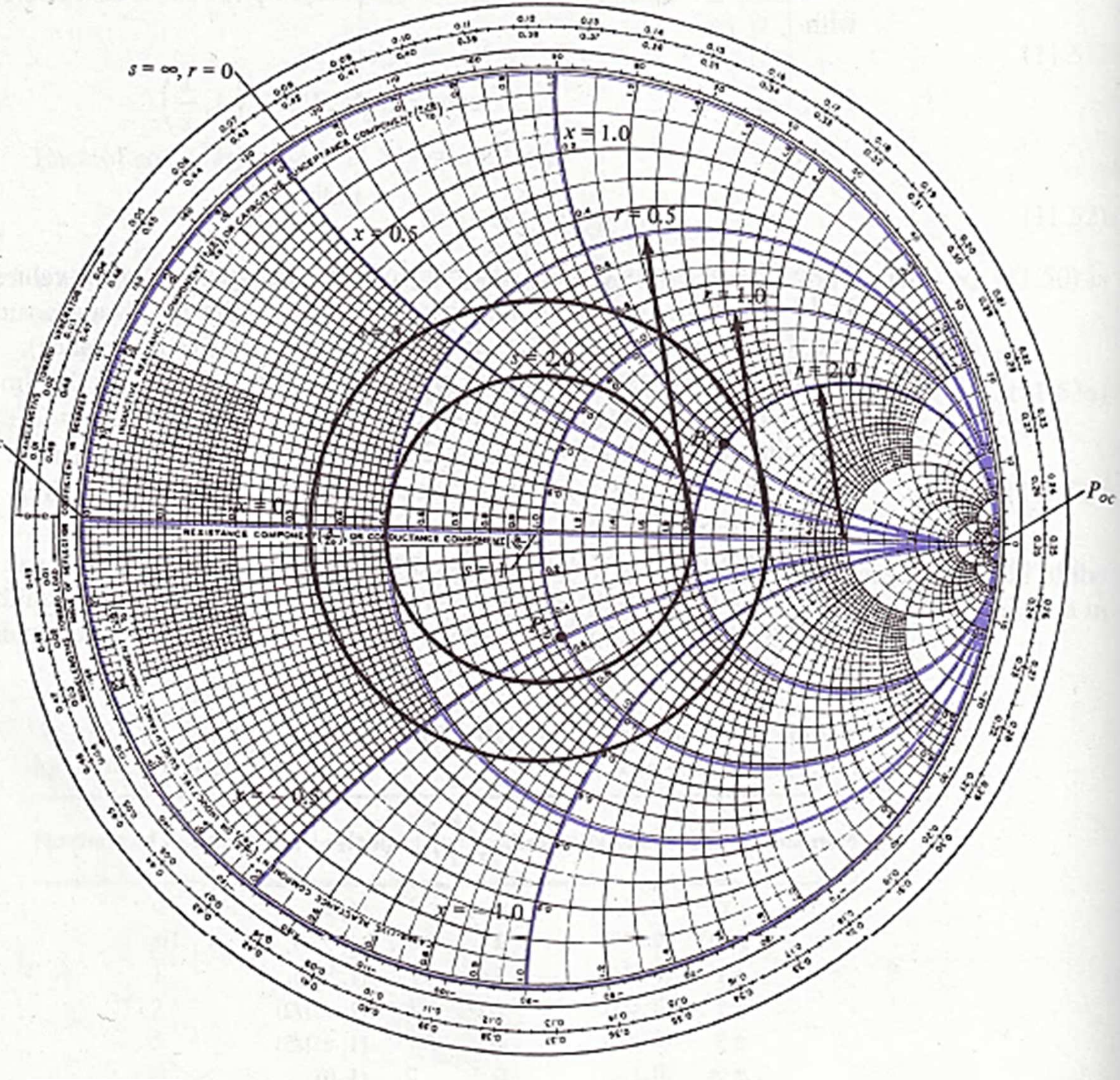


Illustration of the r-, x-, and s-circles on the Smith chart.

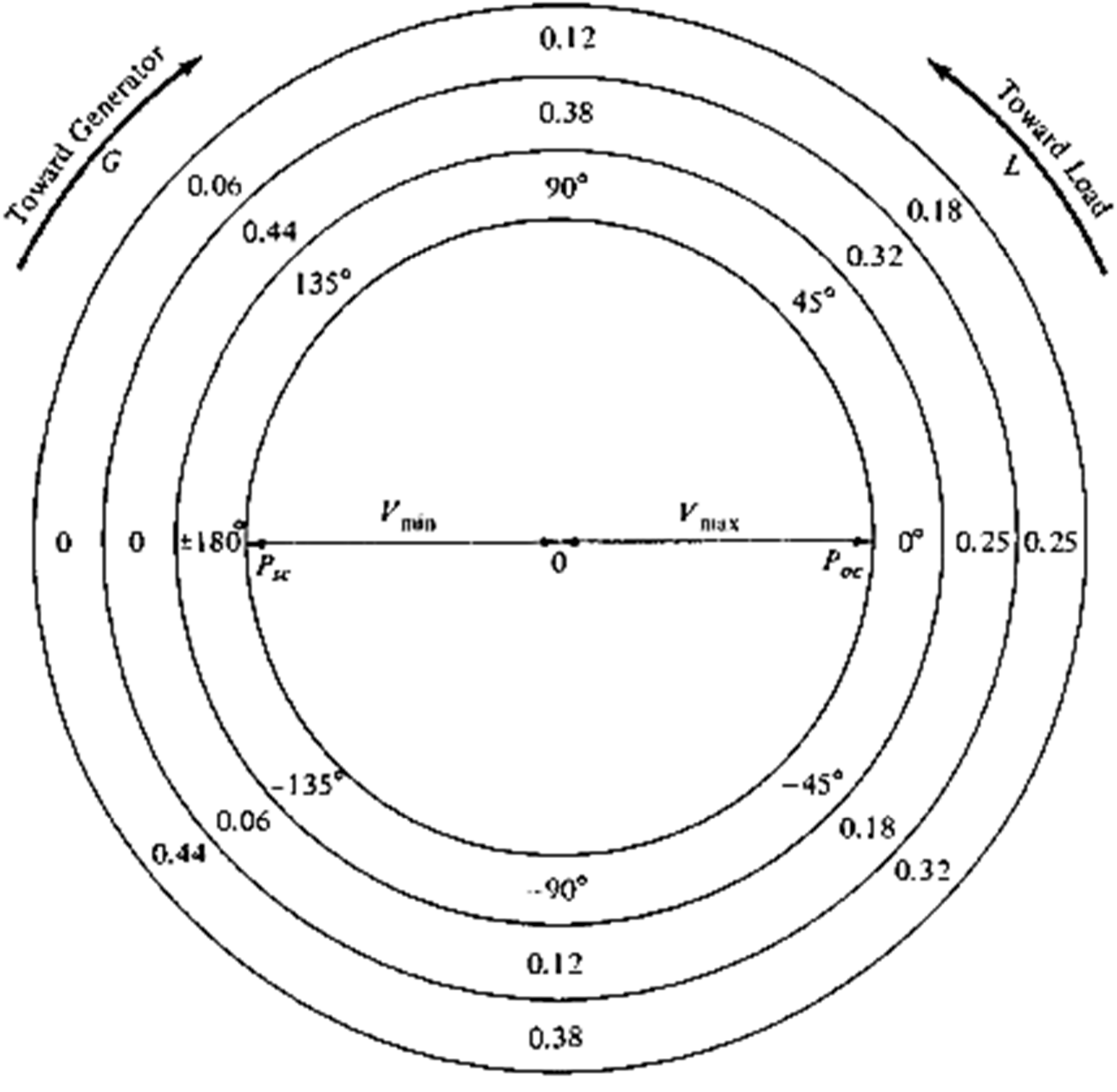
On the chart, locate a normalized impedance $z = 2 + j$, for example, as the point of intersection of the $r = 2$ circle and the $x = 1$ circle. This is point $\boldsymbol{P_1}$

Similarly, $z = 1 - j\,0.5$ is located at $\boldsymbol{P_2}$ where the $r = 1$ circle and the $x = -0.5$ circle intersect.

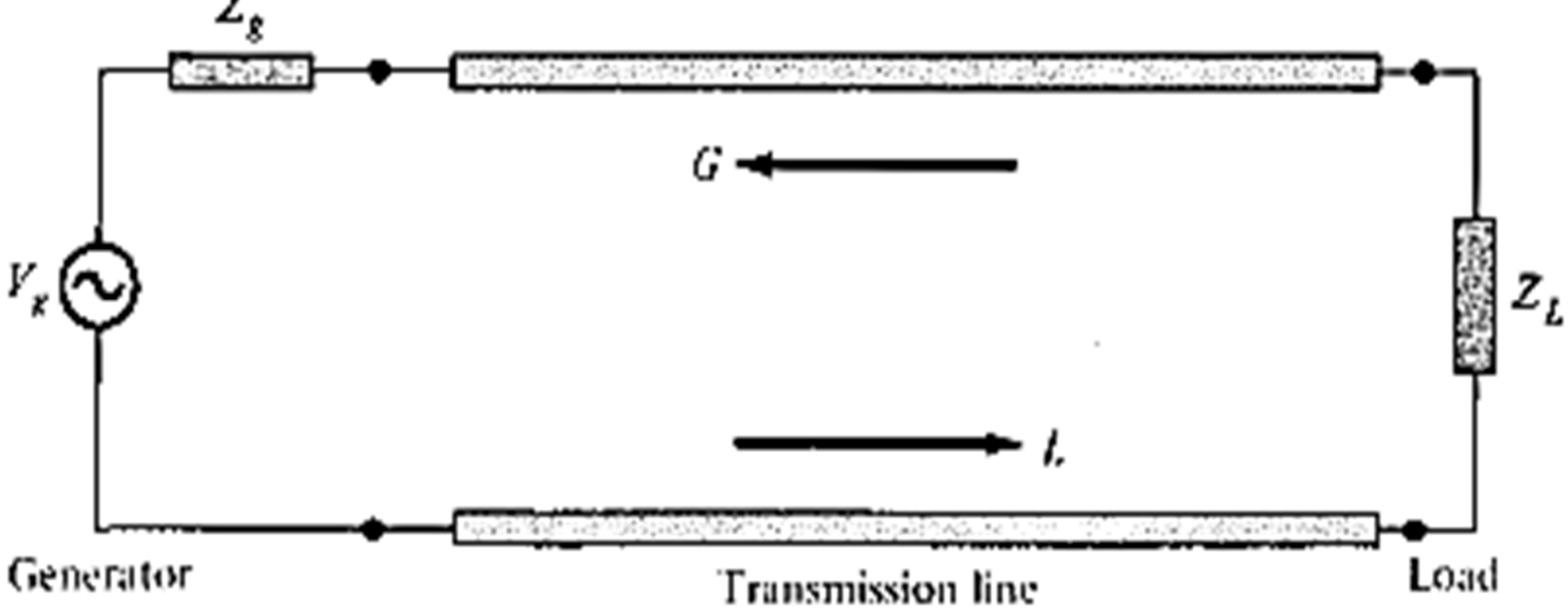
The value of the standing wave ratio *s* is got by locating where an *s*-circle crosses the Γ_r axis.

Points to Ponder : Smith chart

1. At point P_{sc} , $r = 0$, $x = 0$; that is, $Z_L = 0 + j0$ showing that P_{sc} represents a short circuit on the transmission line.
2. At point P_{oc} , $r = \infty$ and $x = \infty$, or $Z_L = \infty + j\,\infty$, which implies that P_{oc} corresponds to an open circuit on the line.
3. A complete revolution (360°) around the Smith chart represents a distance of $\lambda/2$ on the line.
4. Clockwise movement on chart is regarded as moving toward generator (or away from the load) as shown by the arrow G.



Toward Generator
G
Toward Load
L
0.12
0.38
90°
0.06
0.44
135°
0.18
0.32
45°
V_{min}
V_{max}
0
0
±180°
P_{sc}
0
P_{oc}
0°
0.25
0.25
−135°
−45°
0.06
0.44
0.18
0.32
−90°
0.12
0.38



Z_g
G
V_g
Z_L
L
Generator
Transmission line
Load

5. Similarly, counterclockwise movement on the chart corresponds to moving toward the load (or away from the generator) as indicated by the arrow L.
6. Notice that at the load, moving toward the load does not make sense (because we are already at the load).
7. The same can be said of the case when we are at the generator end.
8. There are <u>three scales</u> around the periphery of the Smith chart.

9. The scales are used in determining the distance from the load or generator in degrees or wavelengths.
10. The outermost scale is used to determine the distance on the line from the generator end in terms of wavelengths.
11. The next scale determines the distance from the load end in terms of wavelengths.
12. The innermost scale is a protractor to determine θ_{Γ}; it can also be used to determine distance from load or generator.

13. Since a $\lambda/2$ distance on the line corresponds to a movement of 360° on the chart, a λ distance on the line corresponds to a 720° movement on the chart.
14. V_{max} occurs where $Z_{in,max}$ is located on the chart and that is on the positive Γ_r axis or on OP_{OC}
15. Vmin is located at the same point where we have $Z_{in,min}$ on the chart; that is, on the negative Γ_r axis or on OPsc.

16. Notice that V_{max} and V_{min} (or $Z_{in,max}$ and $Z_{in,min}$) are $\lambda/4$ (or 180°) apart.

17. The Smith chart is used both as impedance chart and admittance chart ($Y = 1/Z$).
18. As admittance chart (normalized impedance $y = Y/Y_0 = g + jb$), the g- and b-circles correspond to r- and x-circles, respectively.

Problems on Smith Chart

A 30-m-long lossless transmission line with $Z_o = 50\ \Omega$ operating at 2 MHz is terminated with a load $Z_L = 60 + j40\ \Omega$. If $u = 0.6c$ on the line, find

(a) The reflection coefficient Γ

(b) The standing wave ratio s

(c) The input impedance

Sol: **Method 1:** (Without the Smith chart)

(a) $$\Gamma = \frac{Z_L - Z_o}{Z_L + Z_o} = \frac{60 + j40 - 50}{50 + j40 + 50} = \frac{10 + j40}{110 + j40} = 0.3523\underline{/56^\circ}$$

(b) $$s = \frac{1 + |\Gamma|}{1 - |\Gamma|} = \frac{1 + 0.3523}{1 - 0.3523} = 2.088$$

(c) Since $u = \omega/\beta$, or $\beta = \omega/u$,

$$\beta\ell = \frac{\omega\ell}{u} = \frac{2\pi\,(2 \times 10^6)(30)}{0.6\,(3 \times 10^8)} = \frac{2\pi}{3} = 120^\circ$$

Note that $\beta\ell$ is the electrical length of the line.

$$
\begin{aligned}
Z_{\text{in}} &= Z_o\left[\frac{Z_L + jZ_o \tan\beta\ell}{Z_o + jZ_L \tan\beta\ell}\right] \\
&= \frac{50\,(60 + j40 + j50\tan 120^\circ)}{[50 + j(60 + j40)\tan 120^\circ]} \\
&= \frac{50\,(6 + j4 - j5\sqrt{3})}{(5 + 4\sqrt{3} - j6\sqrt{3})} = 24.01\angle 3.22^\circ \\
&= 23.97 + j1.35\ \Omega
\end{aligned}
$$

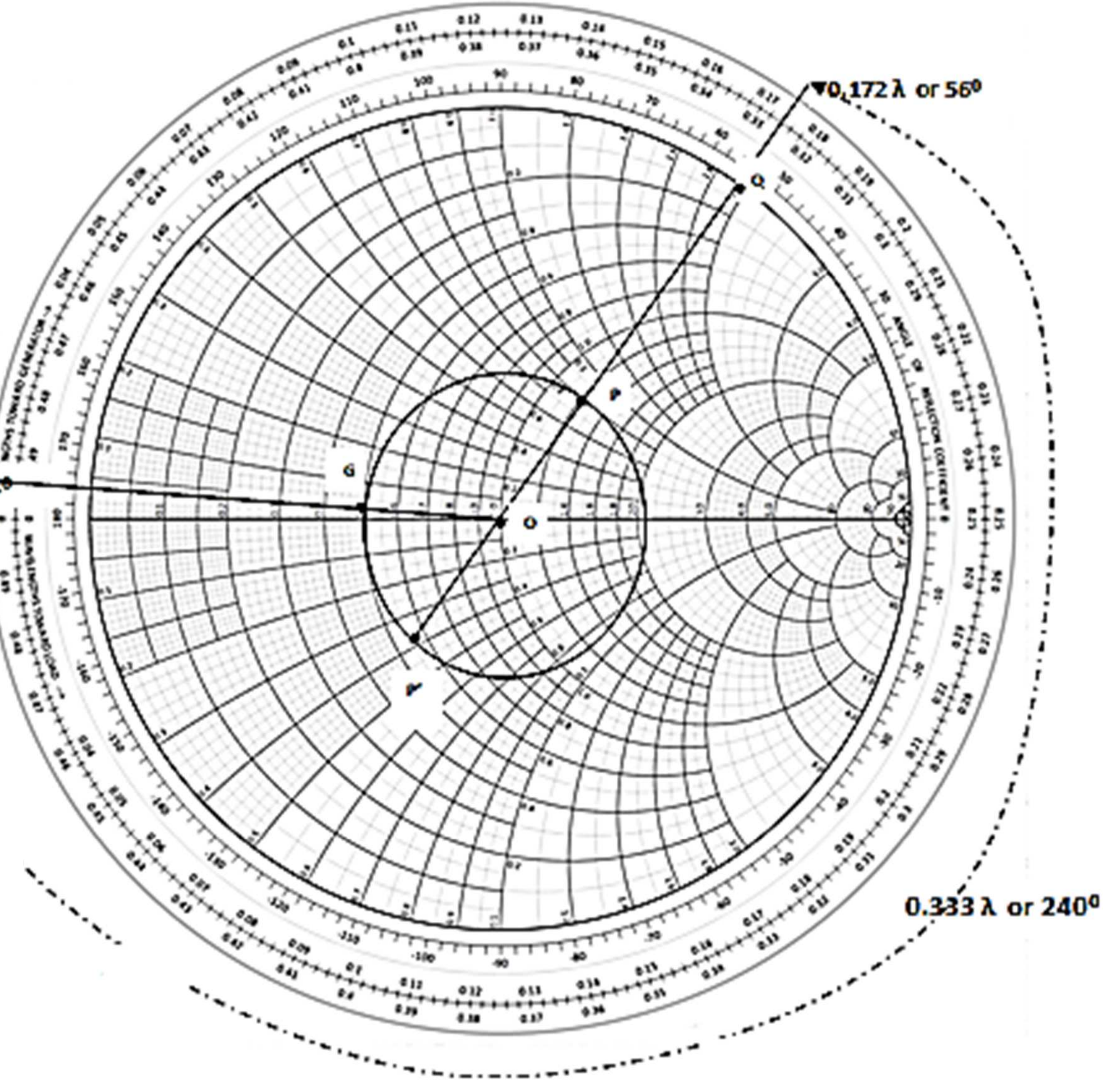
Method 2: (Using the Smith chart).

(a) Calculate the normalized load impedance

$$z_L = \frac{Z_L}{Z_o} = \frac{60 + j40}{50}$$
$$= \underline{1.2 + j0.8}$$

Locate z_L on the Smith chart at point P where the $r = 1.2$ circle and the $x = 0.8$ circle meet. To get Γ at z_L, extend OP to meet the $r = 0$ circle at Q and measure OP and OQ. Since OQ corresponds to $|\Gamma| = 1$, then at P,

$\therefore \quad \Gamma = OP/OQ = 2.7/7.8 = \underline{0.346}$



Z_L = 60+j40 Ω,
Z_o = 50Ω

$P(z_L)$ = 1.2 +j0.8

$P'(y_L)$ = 0.575-j0.39

VSWR = 2.1

Γ =OP/OQ =2.7/7.8 = 0.346

angle Γ = 56^0

$G(z_x)$ = 0.5 +j0.03

$Z_L = 60+j40\ \Omega$, $Z_o = 50\Omega$

$\therefore$ $P(z_L) = 1.2 + j0.8$

$\therefore$ $P'(y_L) = 0.575 - j0.39$

VSWR = $\underline{2.1}$

$\therefore$ Γ =OP/OQ =2.7/7.8 = $\underline{0.346}$

$\therefore$ angle $\Gamma = \underline{56^0}$

$\therefore$ $\lambda = u/f = 0.6 \times 3 \times 10^8 / (2 \times 10^6) = 90$ m

$\therefore$ $\ell = 30$ m $=(30/90)\ \lambda = \lambda/3 = \underline{0.333\ \lambda}$

or $\lambda = 720^o/3 = \underline{240^o}$

<u>Move towards generator in clockwise direction from point P to G exactly 0.333 λ or 240^o away to point G.</u> At G read the value of impedance.

Denormalize to get exact values: $G(z_x) = 0.5 + j0.03$

$z_x = 0.5 + j0.03$

$\therefore$ $Z_x = 50(0.5 + j0.03)$

$= \underline{(25 + j1.5)\Omega}$

$\therefore$ $y_L = 0.575 - j0.39$

$\therefore$ $Y_L = (1/50) \times (0.575 - j0.39)$

$= \underline{(0.0115 - j0.0078)\ S}$

Angle θ_Γ is read directly on the chart as the angle between OS and OP; that is

$$\theta_\Gamma = \text{angle } POS = 56°$$

$$\therefore \quad \Gamma = \underline{0.346 \ \angle 56°}$$

(b) To obtain the standing wave ratio s,

Locate point S where the s-circle meets the Γ_r-axis.

This is the constant s or $|\Gamma|$ circle.

draw a circle with radius OP and center at O.

$$\therefore \quad s = r \ (\text{for } r \geq 1)$$
$$= 2.1$$

APPLICATIONS OF TRANSMISSION LINES

- Transmission lines are specifically used for load matching and impedance measurements.
- **A. Quarter-Wave Transformer (Matching)**
- When $Z_o \neq Z_L$, the load is *mismatched* and a reflected wave exists on the line.
- However, for maximum power transfer, it is desired that the load be matched to the transmission line ($Z_o = Z_L$) so that there is no reflection ($|\Gamma| = 0$ *or* $s = 1$).
- The matching is achieved by using shorted sections of transmission lines.

when $\ell = \lambda/4$ or $\beta\ell = (2\pi/\lambda)(\lambda/4) = \pi/2$,

$$Z_{in} = Z_o\left[\frac{Z_L + jZ_o \tan \pi/2}{Z_o + jZ_L \tan \pi/2}\right] = \frac{Z_o^2}{Z_L}$$

$$\therefore \quad \frac{Z_{in}}{Z_o} = \frac{Z_o}{Z_L}$$

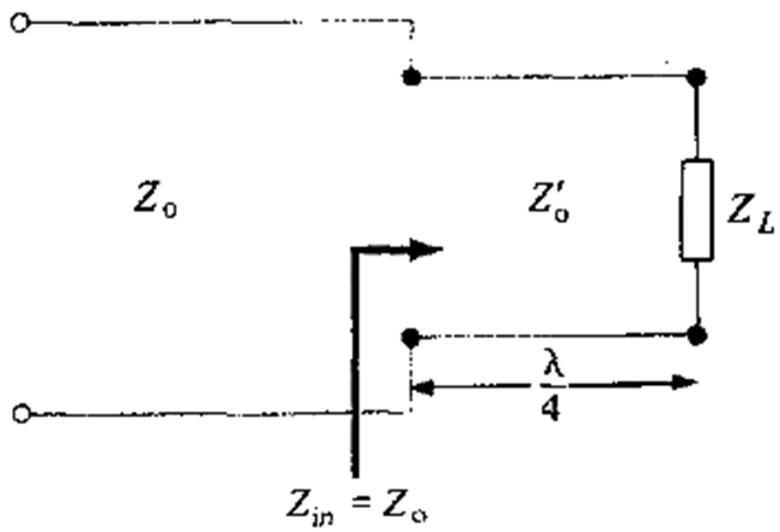
$$\therefore \quad z_{in} = \frac{1}{z_L} \rightarrow y_{in} = z_L$$

Thus by adding a $\lambda/4$ line on the Smith chart, obtain the input admittance corresponding to a given load impedance.

Also, a mismatched load Z_L can be properly matched to a line (with characteristic impedance Z_o) by inserting prior to the load a transmission line $\lambda/4$ long (with characteristic impedance Zo') as shown.

The λ /4 section of the transmission line is called a *quarter-wave transformer* because it is used for impedance matching like an ordinary transformer. Z'_o is selected such that ($Z_{in} = Z_o$)

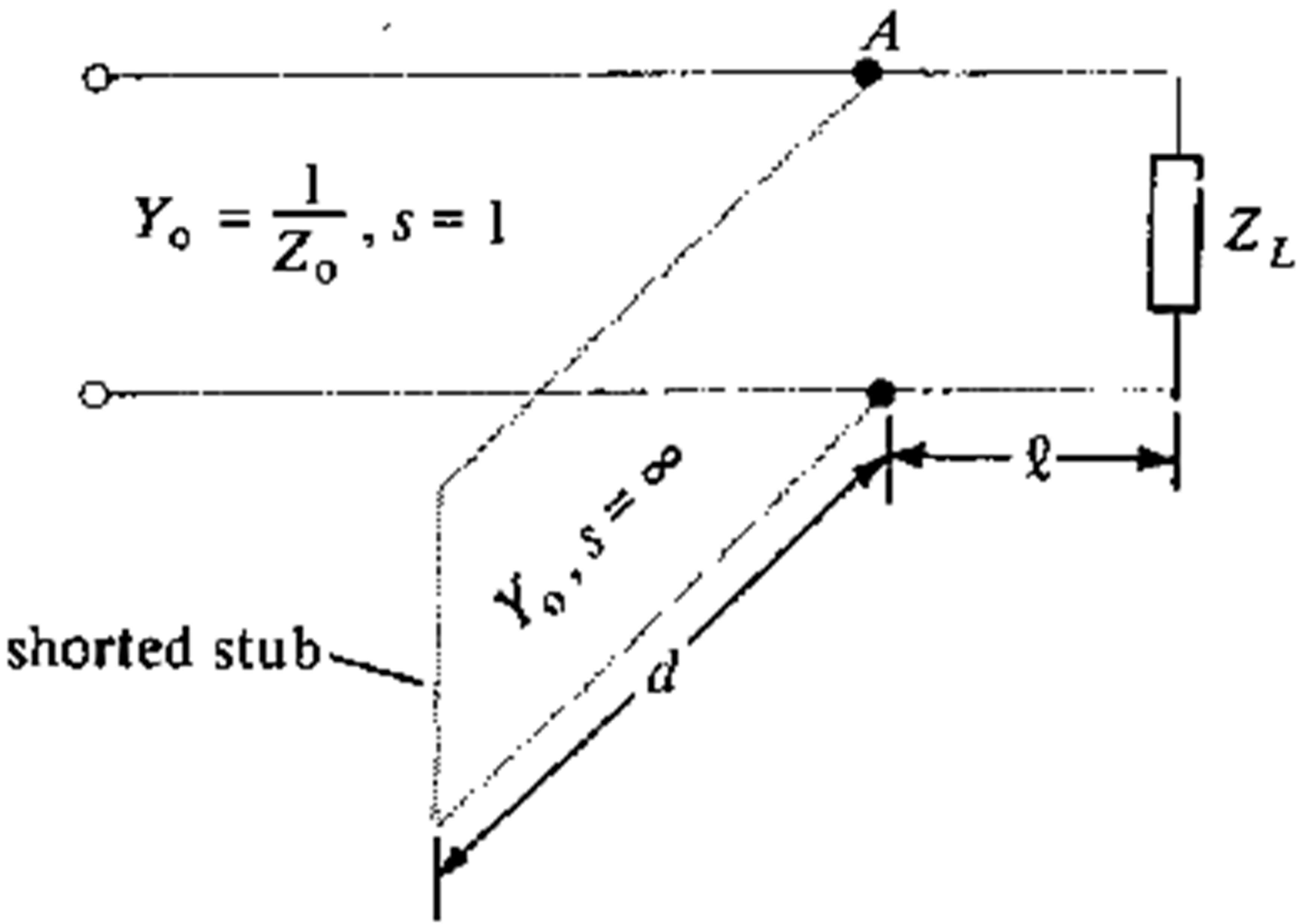
$$Z'_o = \sqrt{Z_o Z_L}$$



Thus, the main disadvantage of the quarter-wave transformer is that it is a narrow-band or frequency-sensitive device.

Single-Stub Tuner (Matching)

- The major drawback of using a quarter-wave transformer as a line-matching device is eliminated by using a *single-stub* tuner.
- The tuner consists of an open or shorted section of transmission line of length *d* connected in parallel with the main line at some distance ℓ from the load as shown.
- Notice that the stub has the same characteristic impedance as the main line.
- It is more difficult to use a series stub although it is theoretically feasible.
- An open-circuited stub radiates some energy at high frequencies.
- As a result, shunt short-circuited parallel stubs are preferred.



Matching with a single-stub tuner.

$Z_{in} = Z_o$, that is, $z_{in} = 1$ or $y_{in} = 1$ at point A on the line

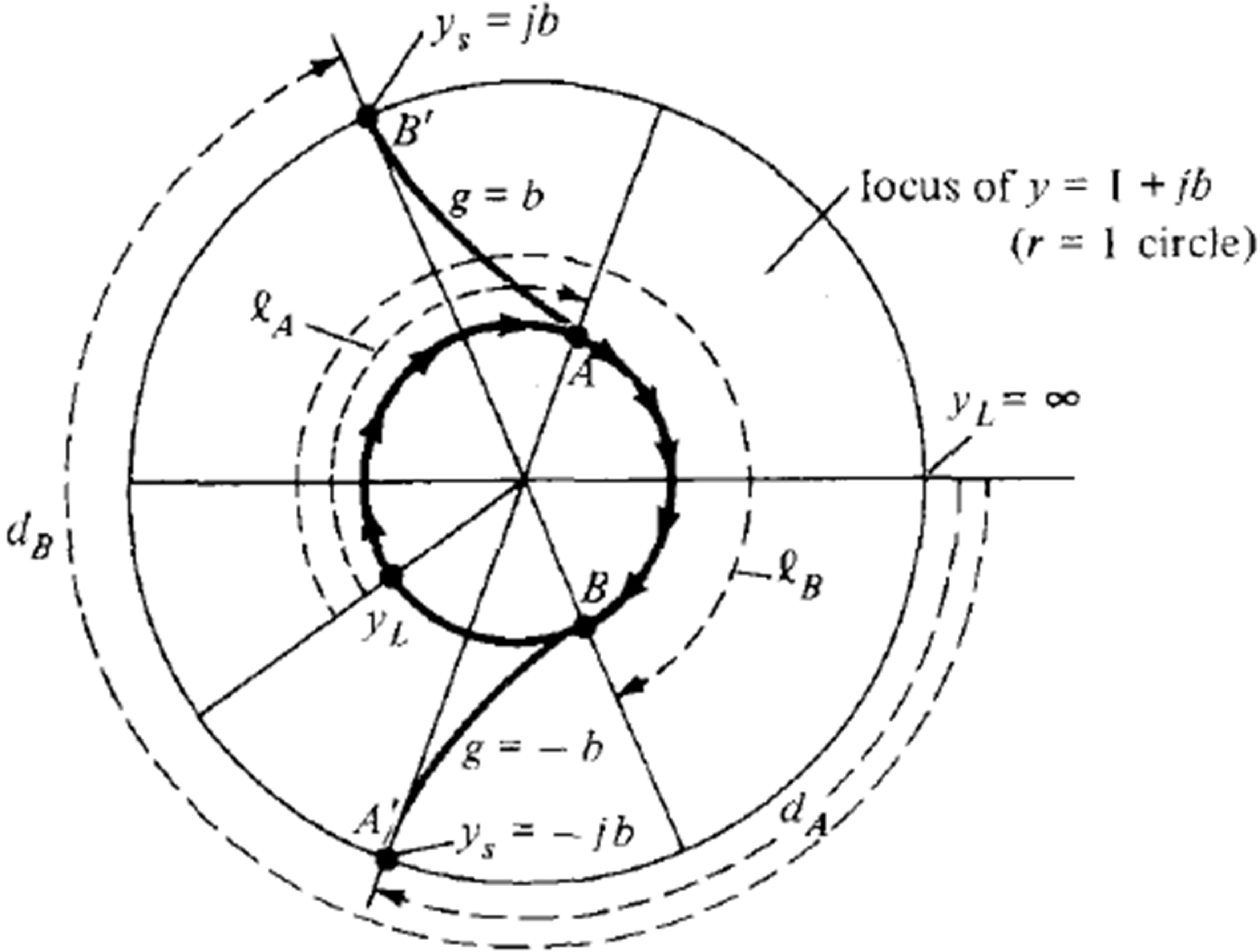
draw the locus $y = 1 + jb$ ($r = 1$ circle) on the Smith chart

shunt stub of admittance $y_s = -jb$ is introduced at A,

$$y_{in} = 1 + jb + y_s = 1 + jb - jb = 1 + j0$$

Since b could be positive or negative, two possible values of ℓ ($<\lambda/2$) can be found on the line.

At A, $y_s = -jb$, $\ell = \ell_A$ and at B, $y_s = jb$, $\ell = \ell_B$



Using the Smith chart to determine *l* and *d* of a shunt-shorted single-stub tuner.

Due to the fact that the stub is shorted($y_L' = \infty$), determine the length d of the stub by finding the distance from P_{sc} (at which $z_L' = 0 + j0$) to the required stub admittance y_s.

For the stub at A, we obtain $d = d_A$ as the distance from P to A', where A' corresponds to $y_s = —jb$ located on the periphery of the chart.

Similarly, obtain $d = d_B$ as the distance from P_{sc} to B' ($y_s = jb$).

Obtain $d = d_A$ and $d = d_B$ corresponding to A and B, respectively, as shown.

Note that $d_A + d_B = \lambda/2$ always.

Since there are two possible shunted stubs, choose to match the shorter stub or one at a position closer to the load.

Instead of having a single stub shunted across the line, we may have two stubs.

This is called *double-stub matching* and allows for the adjustment of the load impedance.

Problems on Stub Matching

- A 100 Ω lossless transmission line is to be matched to a load of 100-j80Ω, utilizing a shorted stub assuming an operating frequency of 20MHz and wave velocity of 0.6c, where c is the speed of light in vacuum.
- Make use of the attached **Smith Chart** and carefully estimate the *Voltage Reflection Coefficient*, *Voltage Standing Wave Ratio*, *Load Admittance*, *Stub length*, *Distance of Stub from the load* and the required *Stub Admittance*.
- Show the working steps in arriving at solution.

Antenna with impedance $40 + j30\ \Omega$ is to be matched to a 100-Ω lossless line with a shorted stub. Determine

(a) The required stub admittance

(b) The distance between the stub and the antenna

(c) The stub length

(d) The standing wave ratio on each ratio of the system

Sol

$$\text{(a)}\quad z_L = \frac{Z_L}{Z_o} = \frac{40 + j30}{100} = 0.4 + j0.3$$

Locate z_L on the Smith chart draw the s-circle

$$\therefore\ y_L = 1.6 - j1.2.$$

Or $$y_L = \frac{Z_o}{Z_L} = \frac{100}{40 + j30} = 1.6 - j1.2$$

Locate points A and B where the s-circle intersects the $g = 1$ circle.

$$\text{At } A,\ y_s = -j1.04$$
$$\text{at } B,\ y_s = +j1.04.$$

Thus the required stub admittance is

$$Y_s = Y_o y_s = \pm j1.04 \frac{1}{100} = \pm j10.4 \text{ mS}$$

(b) determine the distance between the load (antenna in this case) y_L and the stub.

At A,

$$\ell_A = \frac{\lambda}{2} - \frac{(62° - -39°)\lambda}{720°} = 0.36\lambda$$

At B:

$$\ell_B = \frac{(62^\circ - 39^\circ)}{720^\circ} = 0.032\lambda$$

(c) Locate points *A'* and *B'* corresponding to stub admittance —j1.04 and j1.04, respectively. Determine the stub length (distance from Psc to A' and *B')*

$$d_A = \frac{88^\circ}{720^\circ}\lambda = 0.1222\lambda$$

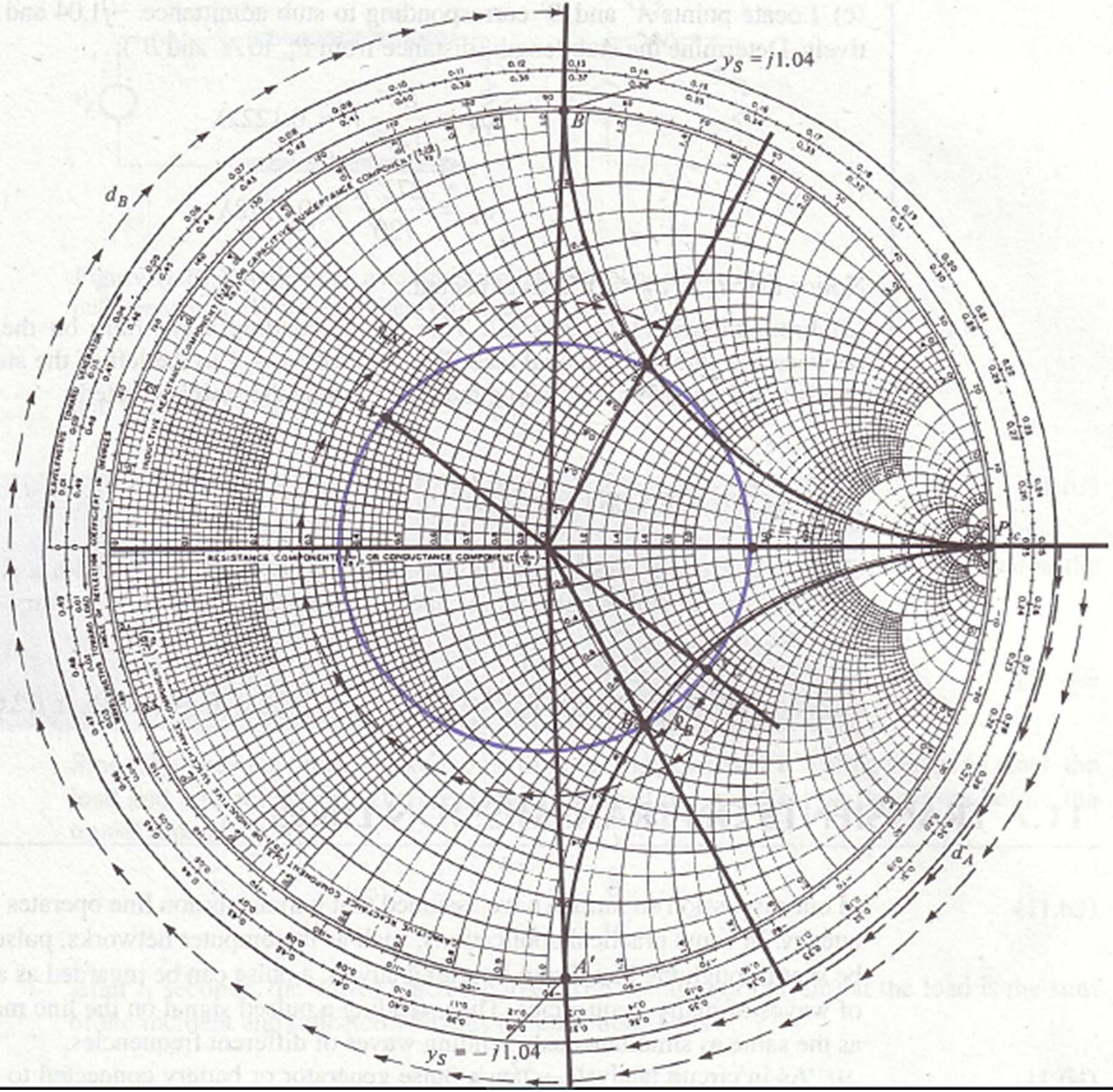
$$d_B = \frac{272^\circ\lambda}{720^\circ} = 0.3778\lambda$$

Notice that $d_A + d_B = 0.5\lambda$ as expected

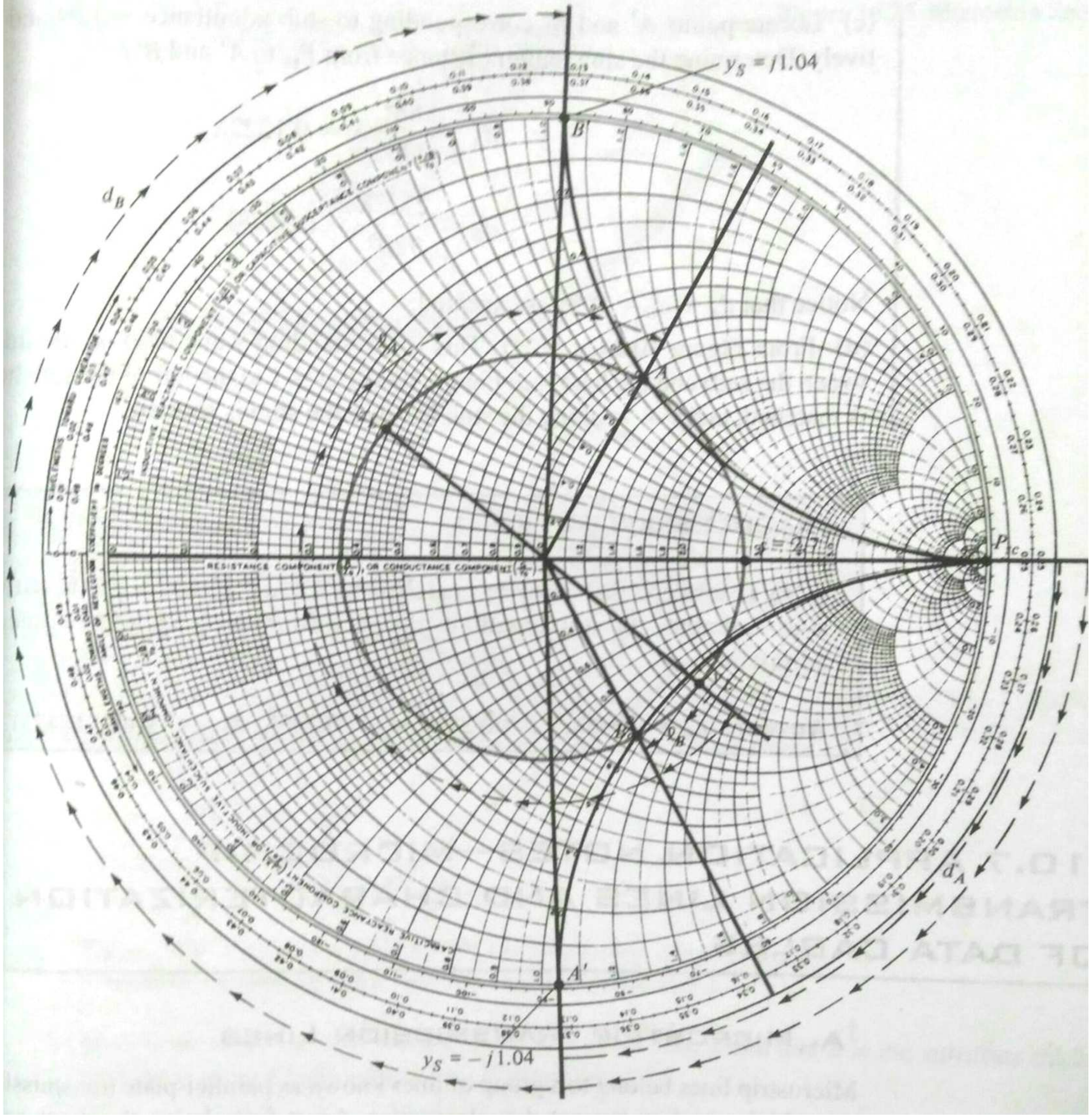
(d) From Smith Chart, $\underline{s = 2.7}$

This is the standing wave ratio on the line segment between the stub and the load $\underline{s = 1}$ to the left of the stub because the line is matched,

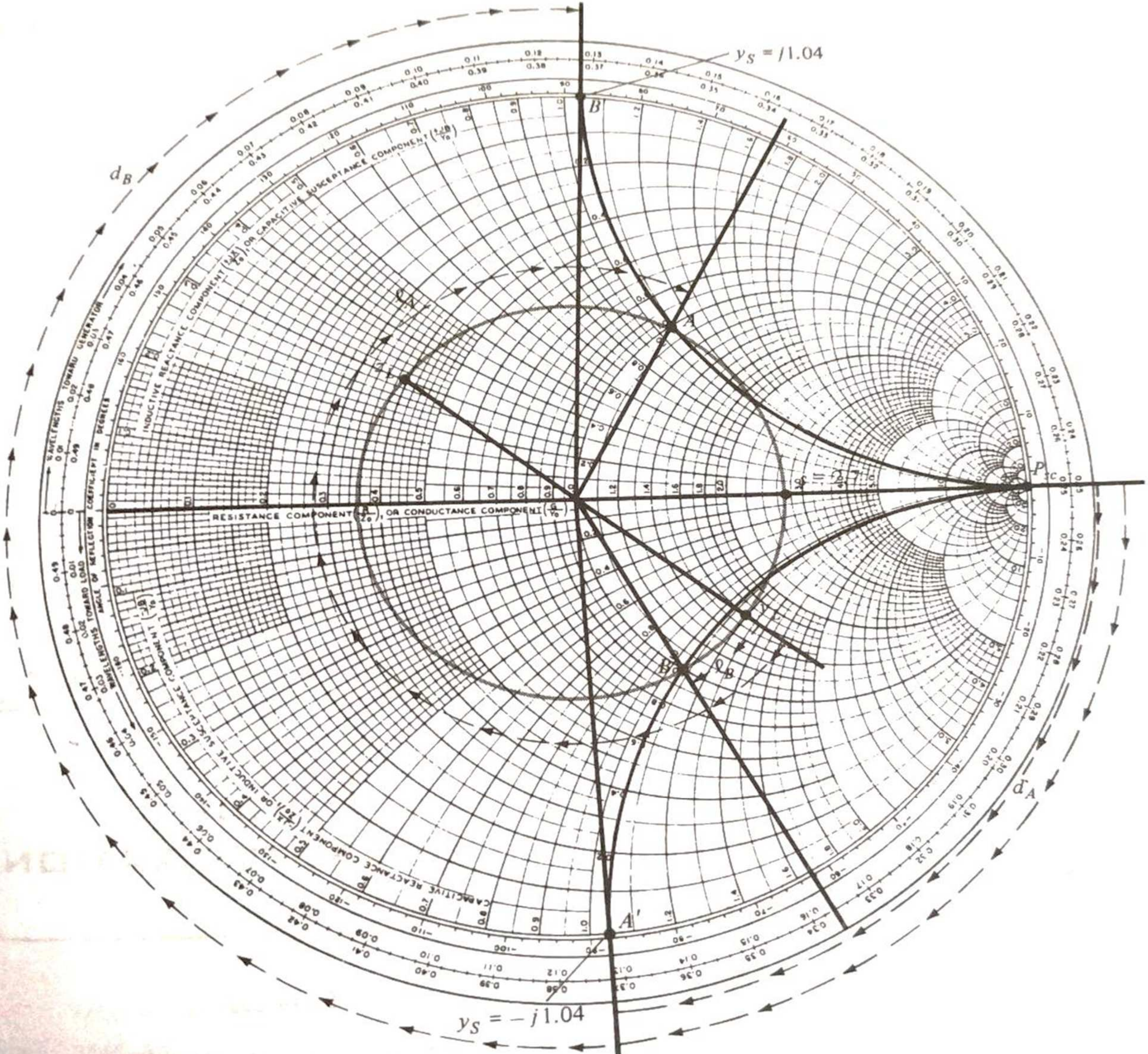
and $\underline{s = \infty}$ along the stub because the stub is shorted at C



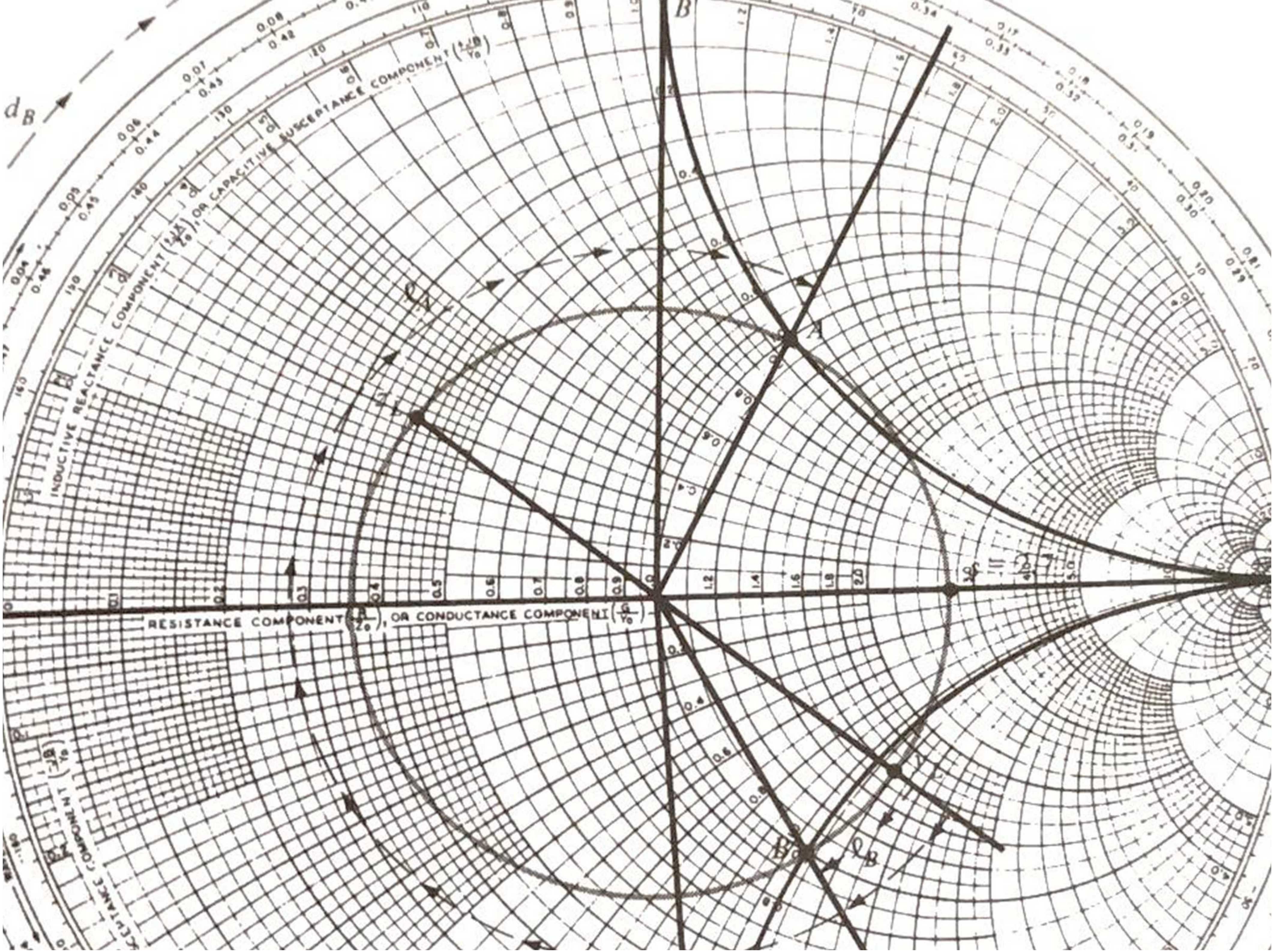
$y_S = j1.04$
B
d_B
A
P_{sc}
RESISTANCE COMPONENT $\left(\frac{R}{Z_0}\right)$, OR CONDUCTANCE COMPONENT $\left(\frac{G}{Y_0}\right)$
B'
d_A
A'
$y_S = -j1.04$



$y_S = j1.04$
B'
d_B
A
P
A'
d_A
$y_S = -j1.04$
RESISTANCE COMPONENT, OR CONDUCTANCE COMPONENT
CAPACITIVE REACTANCE COMPONENT



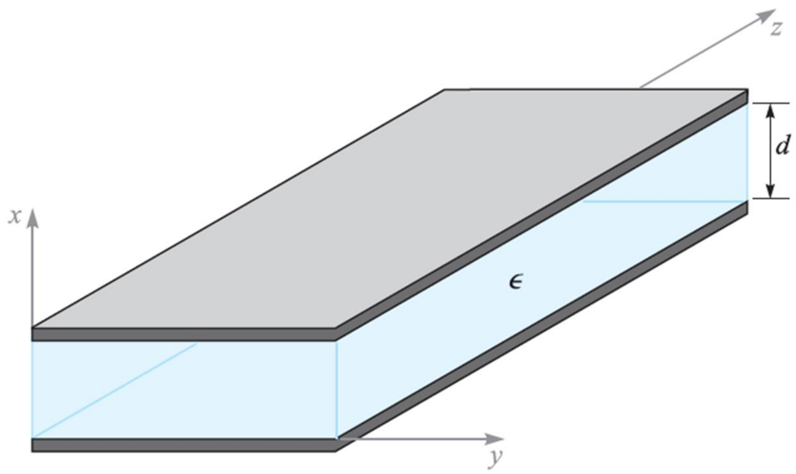
y_S = j1.04
B′
A
P_sc
B
A′
d_B
d_A
y_S = −j1.04
RESISTANCE COMPONENT (R/Z0), OR CONDUCTANCE COMPONENT (G/Y0)
INDUCTIVE REACTANCE COMPONENT (+jX/Z0), OR CAPACITIVE SUSCEPTANCE COMPONENT (+jB/Y0)
CAPACITIVE REACTANCE COMPONENT (−jX/Z0), OR INDUCTIVE SUSCEPTANCE COMPONENT (−jB/Y0)
WAVELENGTHS TOWARD GENERATOR
WAVELENGTHS TOWARD LOAD
ANGLE OF REFLECTION COEFFICIENT IN DEGREES



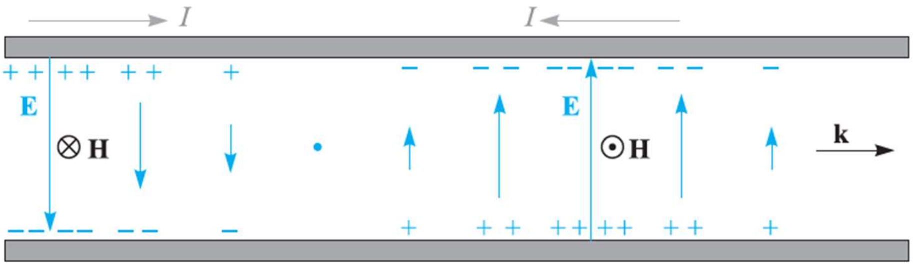
RESISTANCE COMPONENT $\left(\frac{R}{Z_0}\right)$, OR CONDUCTANCE COMPONENT $\left(\frac{G}{Y_0}\right)$
INDUCTIVE REACTANCE COMPONENT $\left(\frac{+jX}{Z_0}\right)$, OR CAPACITIVE SUSCEPTANCE COMPONENT $\left(\frac{+jB}{Y_0}\right)$
d_B
ℓ_A
ℓ_B
A
B

Parallel Plate Waveguide-
TE and TM Waves

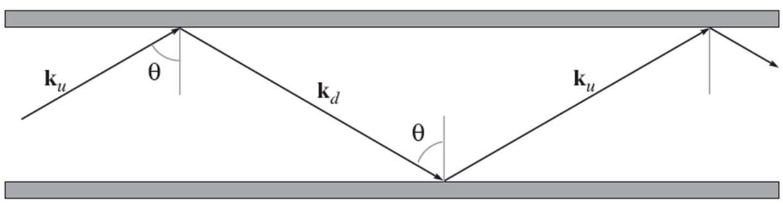
EC 303 Module VI



Parallel-plate waveguide, with metal plates at $x = 0, d$. Between the plates is a dielectric of permittivity ϵ.



A transmission-line wave represented by voltage and current distributions along the length is associated with transverse electric and magnetic fields, forming a TEM wave.



In a parallel-plate waveguide, plane waves can propagate by oblique reflection from the conducting walls. This produces a waveguide mode that is not TEM.

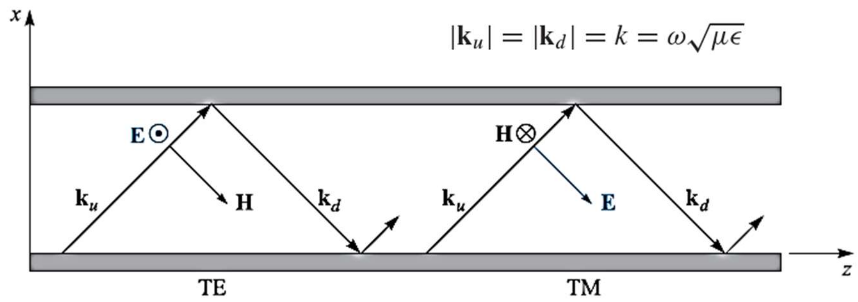
Parallel-Plate Waveguide

- In the parallel-plate guide, two types of waveguide modes can be supported.
- identify a *Transverse Electric* or *TE* mode when **E** is perpendicular to the plane of incidence
- This positions **E** parallel to the transverse plane of the waveguide, as well as to the boundaries.
- Similarly, a *Transverse Magnetic* or *TM* mode when the entire **H** field is in the *y* direction and is thus within the transverse plane of the guide.

The wavevector **k**, indicates the direction of wave travel as well as direction of power flow.

Wavevectors $\mathbf{k}_u$ and $\mathbf{k}_d$ are associated with the upward-and downward-propagating waves, respectively, and these have identical magnitudes,

$$|\mathbf{k}_u| = |\mathbf{k}_d| = k = \omega\sqrt{\mu\epsilon}$$



Plane wave representation of TE and TM modes in a parallel-plate guide.

- Note, for example, that with **E** in the *y* direction (TE mode), **H** will have *x* and *z* components.
- Likewise, a TM mode will have *x* and *z* components of **E**.
- it is not possible to achieve a purely TEM mode for values of θ other than 90◦.
- Other wave polarizations are possible that lie between the TE and TM cases, but these can always be expressed as superposition of TE and TM modes.

PARALLEL-PLATE GUIDE ANALYSIS USING THE WAVE EQUATION

- Analysis of the Waveguide is by use of the Wave Equation, which need to be solved subject to the boundary conditions at the conducting walls.

$$\nabla^2 \mathbf{E}_s = -k^2 \mathbf{E}_s \quad \text{.........(1)}$$

where $k = n\omega/c$

- In TE modes , there is only a *y* component of **E**.
- The wave equation becomes:

$$\therefore \quad \frac{\partial^2 E_{ys}}{\partial x^2} + \frac{\partial^2 E_{ys}}{\partial y^2} + \frac{\partial^2 E_{ys}}{\partial z^2} + k^2 E_{ys} = 0 \quad \text{.........(2)}$$

- We assume that the width of the guide (in the *y* direction) is very large compared to the plate separation *d*.
- Therefore we can assume no *y* variation in the fields (fringing fields are ignored), and so,

$$\therefore \quad \partial^2 E_{ys}/\partial y^2 = 0.$$

$$\therefore \quad E_{ys} = E_0 f_m(x) e^{-j\beta_m z} \quad \text{.........(3)}$$

- where E_0 is a constant, and where $f_m(x)$ is a normalized function to be determined (whose maximum value is unity).

- We now substitute (3) into (2) to obtain,

$$\therefore \quad \frac{d^2 f_m(x)}{dx^2} + \left(k^2 - \beta_m^2\right) f_m(x) = 0 \quad \text{.........(4)}$$

- The general solution is,

$$\therefore \; f_m(x) = \cos(\kappa_m x) + \sin(\kappa_m x) \quad \text{.........(5)}$$

next apply the appropriate boundary conditions to evaluate κ_m.

conducting boundaries appear at $x = 0$ and $x = d$, at which the tangential electric field (E_y) must be zero.

In Eq. (5), only the sin($\kappa_m x$) term will allow the boundary conditions to be satisfied, so keep it and drop the cosine term.

The x = 0 condition is automatically satisfied by the sine function.

The x = d condition is met when we choose the value of κ_m such that,

$$\kappa_m = \frac{m\pi}{d} \quad \text{.........(6)}$$

The final form of E_{ys} is obtained by substituting $f_m(x)$ as expressed through (5) and (6) into (3),

$$\therefore \quad E_{ys} = E_0 \sin\left(\frac{m\pi x}{d}\right) e^{-j\beta_m z} \quad \text{.........(7)}$$

- An additional significance of the mode number m is seen when considering the form of electric field of (7).
- Specifically, m is the number of spatial half-cycles of electric field that occur over the distance d in the transverse plane.
- This can be understood physically by considering the behavior of the guide at cutoff.
- the plane wave angle of incidence in the guide at cutoff is zero, meaning that the wave simply bounces up and down between the conducting walls.

- The wave must be resonant in the structure, so the net round trip phase shift is $2m\pi$.
- With the plane waves oriented vertically, $\theta_m = 0$, and so $\kappa_m = k = 2n\pi/\lambda_{cm}$.
- So at cutoff,

$$\frac{m\pi}{d} = \frac{2n\pi}{\lambda_{cm}} \quad \text{.........(7)}$$

$$\therefore \quad d = \frac{m\lambda_{cm}}{2n} \quad \text{at cutoff} \quad \text{.........(8)}$$

$$\therefore \quad E_{ys} = E_0 \sin\left(\frac{m\pi x}{d}\right) = E_0 \sin\left(\frac{2n\pi x}{\lambda_{cm}}\right) \quad \text{.........(9)}$$

- The waveguide is simply a one-dimensional *resonant cavity,* in which a wave can oscillate in the *x* direction if its wavelength as measured in the medium is an integer multiple of $2d$ where the integer is m.
- Having found the electric field, find the magnetic field using Maxwell's equations.
- obtain *x, z* components of $\mathbf{H}_s$ for a TE mode.
- use the Maxwell equation,

$$\nabla \times \mathbf{E}_s = -j\omega\mu\mathbf{H}_s \quad \text{.........(10)}$$

- where, in the present case of having only a *y* component of $\mathbf{E}_s$,

$$\therefore \quad \nabla \times \mathbf{E}_s = \frac{\partial E_{ys}}{\partial x}\mathbf{a}_z - \frac{\partial E_{ys}}{\partial z}\mathbf{a}_x$$

$$= \kappa_m E_0 \cos(\kappa_m x) e^{-j\beta_m z}\mathbf{a}_z + j\beta_m E_0 \sin(\kappa_m x) e^{-j\beta_m z}\mathbf{a}_x$$

………(11)

Solve for $\mathbf{H}_s$ by div both sides of (10) by $-j\omega\mu$.

Performing this operation on (11), we obtain the two magnetic field components:

$$\therefore \quad H_{xs} = -\frac{\beta_m}{\omega\mu} E_0 \sin(\kappa_m x) e^{-j\beta_m z} \quad \text{.........(12)}$$

$$\therefore \quad H_{zs} = j\frac{\kappa_m}{\omega\mu} E_0 \cos(\kappa_m x) e^{-j\beta_m z} \quad \text{.........(13)}$$

$$\therefore \quad |\mathbf{H}_s| = \sqrt{\mathbf{H}_s \cdot \mathbf{H}_s^*} = \sqrt{H_{xs}H_{xs}^* + H_{zs}H_{zs}^*} \quad \text{.........(14)}$$

$$\therefore \quad |\mathbf{H}_s| = \frac{E_0}{\omega\mu}\left(\kappa_m^2 + \beta_m^2\right)^{1/2}\left(\sin^2(\kappa_m x) + \cos^2(\kappa_m x)\right)^{1/2} \quad \text{.........(15)}$$

$$\because \quad \kappa_m^2 + \beta_m^2 = k^2$$

$$\therefore \quad |\mathbf{H}_s| = \frac{k}{\omega\mu} E_0 = \frac{\omega\sqrt{\mu\epsilon}}{\omega\mu} = \frac{E_0}{\eta} \qquad \text{.........(16)}$$

where $\eta = \sqrt{\mu/\epsilon}$.

define the radian *cutoff frequency* for mode *m* as

$$\omega_{cm} = \frac{m\pi c}{nd}$$

$$\beta_m = \frac{2\pi n}{\lambda}\sqrt{1 - \left(\frac{\lambda}{\lambda_{cm}}\right)^2}$$

$$\kappa_m = \frac{m\pi}{d}$$

$$\theta_m = \cos^{-1}\left(\frac{m\pi}{kd}\right) = \cos^{-1}\left(\frac{m\pi c}{\omega n d}\right) = \cos^{-1}\left(\frac{m\lambda}{2nd}\right)$$

$$\beta_m = \sqrt{k^2 - \kappa_m^2} = k\sqrt{1 - \left(\frac{m\pi}{kd}\right)^2} = k\sqrt{1 - \left(\frac{m\pi c}{\omega n d}\right)^2}$$

$$\beta_m = \frac{n\omega}{c}\sqrt{1 - \left(\frac{\omega_{cm}}{\omega}\right)^2}$$

$$\lambda_{cm} = \frac{2\pi c}{\omega_{cm}} = \frac{2nd}{m}$$

A parallel-plate waveguide has plate separation d = 1 cm and is filled with teflon having dielectric constant r = 2.1. Determine the maximum operating frequency such that only the TEM mode will propagate. Also find the range of frequencies over which the TE1 and TM1 (m = 1) modes, and no higher-order modes, will propagate.

Sol

the cutoff frequency for the first waveguide mode (m = 1)

$$f_{c1} = \frac{\omega_{c1}}{2\pi} = \frac{2.99 \times 10^{10}}{2\sqrt{2.1}} = 1.03 \times 10^{10}\ \text{Hz} = 10.3\ \text{GHz}$$

To propagate only TEM waves, we must have $f < 10.3$ GHz. To allow TE_1 and TM_1 (along with TEM) only, the frequency range must be $\omega_{c1} < \omega < \omega_{c2}$, where $\omega_{c2} = 2\omega_{c1}$, from (41). Thus, the frequencies at which we will have the $m = 1$ modes and TEM will be $10.3\,\text{GHz} < f < 20.6\,\text{GHz}$.

EXAMPLE 2

In the parallel-plate guide of Example 1, the operating wavelength is λ = 2 mm. How many waveguide modes will propagate?

Sol

For mode m to propagate, the requirement is$\lambda < \lambda cm$. For the given waveguide
and wavelength, the inequality becomes,

$$2\ \text{mm} < \frac{2\sqrt{2.1}\ (10\,\text{mm})}{m}$$

$$m < \frac{2\sqrt{2.1}\ (10\,\text{mm})}{2\,\text{mm}} = 14.5$$

Thus the guide will support modes at the given wavelength up to order m = 14. Since there will be a TE and a TM mode for each value of m, this gives, not including the TEM mode, a total of 28 guided modes that are above cutoff.

$$\cos\theta_m = \frac{\omega_{cm}}{\omega} = \frac{\lambda}{\lambda_{cm}}$$

$$\beta_m = k\sin\theta_m = \frac{n\omega}{c}\sin\theta_m$$

phase velocity of mode *m* $$v_{pm} = \frac{\omega}{\beta_m} = \frac{c}{n\sin\theta_m}$$

energy will propagate at the group velocity, v_g

$$v_{gm} = \frac{c}{n}\sqrt{1-\left(\frac{\omega_{cm}}{\omega}\right)^2} = \frac{c}{n}\sin\theta_m$$

EXAMPLE 3

In the guide of Example 1, the operating frequency is 25 GHz. Consequently, modes for which $m = 1$ and $m = 2$ will be above cutoff. Determine the *group delay difference* between these two modes over a distance of 1 cm. This is the difference in propagation times between the two modes when energy in each propagates over the 1-cm distance.

Sol

The group delay difference is expressed as

$$\Delta t = \left(\frac{1}{v_{g2}} - \frac{1}{v_{g1}}\right) \text{ (s/cm)}$$

$$v_{g1} = \frac{c}{\sqrt{2.1}}\sqrt{1-\left(\frac{10.3}{25}\right)^2} = 0.63c$$

$$v_{g2} = \frac{c}{\sqrt{2.1}}\sqrt{1-\left(\frac{20.6}{25}\right)^2} = 0.39c$$

$$\Delta t = \frac{1}{c}\left[\frac{1}{.39} - \frac{1}{.63}\right] = 3.3 \times 10^{-11} \text{ s/cm} = 33 \text{ ps/cm}$$

This computation gives a rough measure of the *modal dispersion* in the guide, applying to the case of having only two modes propagating.

A pulse, for example, whose center frequency is 25 GHz would have its energy divided between the two modes.

The pulse would broaden by approximately 33 ps/cm of propagation distance as the energy in the modes separates.

If, however, we include the TEM mode (as we really must), then the broadening will be even greater.

The group velocity for TEM will be $c/\sqrt{2.1}$. The group delay difference of interest will then be between the TEM mode and the m = 2 mode (TE or TM). We would therefore have,

$$\Delta t_{\text{net}} = \frac{1}{c}\left[\frac{1}{.39} - 1\right] = 52 \text{ ps/cm}$$

Simple numerical calculations on wave guides – Please work it out!

Determine the wave angles θ_m for the first four modes ($m = 1, 2, 3, 4$) in a parallel-plate guide with $d = 2$ cm, $\epsilon'_r = 1$, and $f = 30$ GHz.

A parallel-plate guide has plate spacing $d = 5$ mm and is filled with glass ($n = 1.45$). What is the maximum frequency at which the guide will operate in the TEM mode only?

A parallel-plate guide having $d = 1$ cm is filled with air. Find the cutoff wavelength for the $m = 2$ mode (TE or TM).

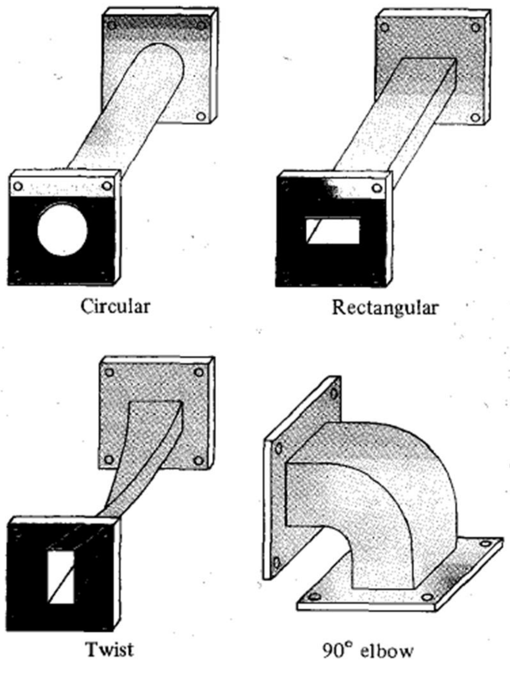
The Hollow Rectangular Waveguide

EC303 AET Module- VI – Part-2

Introduction

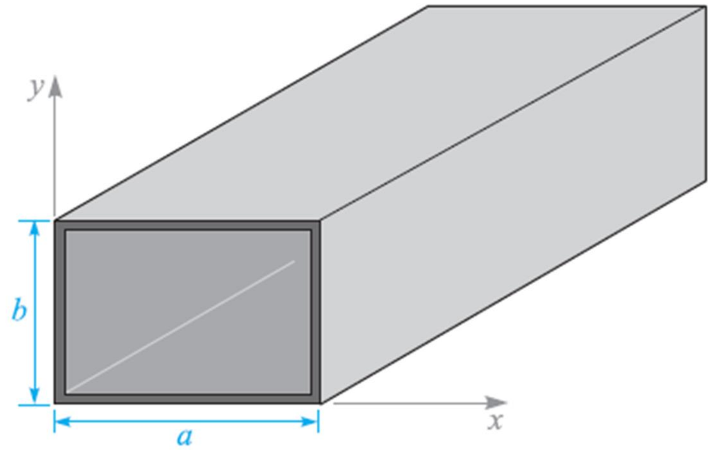
- A transmission line can be used to guide EM energy from one point (generator) to another (load).
- A waveguide is another way to achieve the same goal.
- Transmission line can support only a transverse electromagnetic (TEM) wave, whereas a waveguide can support many possible field configurations.
- At microwave frequencies (roughly 3-300 GHz), transmission lines become inefficient due to Skin Effect and Dielectric Losses.
- Hollow Waveguides are used at that range of frequencies to obtain larger bandwidth and lower signal attenuation.

- A transmission line may operate from dc (f = 0) to a very high frequency.
- A waveguide can operate only above a certain frequency called the *cutoff frequency* and therefore acts as a high-pass filter.
- Thus, waveguides cannot transmit dc, and they become excessively large at frequencies below microwave frequencies.



Typical waveguides.

RECTANGULAR WAVEGUIDES

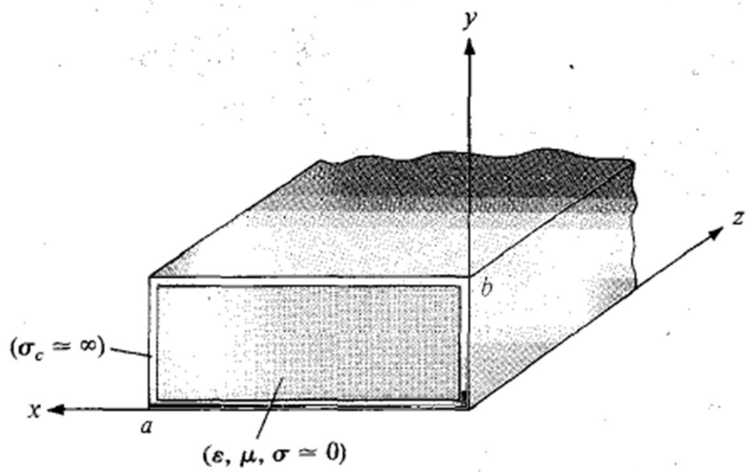


- The propagation direction is along the *z* axis.
- The guide is of width *a* along *x* and height *b* along *y*.
- relate the geometry to that of the parallel-plate guide by thinking of the rectangular guide as two parallel-plate guides of orthogonal orientation that are assembled to form one unit.
- a pair of horizontal conducting walls (along the *x* direction) and a pair of vertical walls (along *y*), all of which form one continuous boundary.

- In the parallel-plate guide, the TEM mode can exist, along with TE and TM modes.
- The rectangular guide will support the TE and TM modes, but it *will not support a TEM mode*.
- Unlike the parallel-plate guide, there's a conducting boundary that completely surrounds the transverse plane.
- The nonexistence of TEM can be understood by recalling that any electric field must have a zero tangential component at the boundary.
- This means that it is impossible to set up an electric field that will not exhibit the sideways variation that is necessary to satisfy this boundary condition.

- Because **E** varies in the transverse plane, the computation of **H** through $\nabla \times \mathbf{E} = -j\omega\mu\mathbf{H}$ *must* lead to a *z* component of **H**, and so cannot have a TEM mode.
- Cannot find any other orientation of a completely transverse **E** in the guide that will allow a completely transverse **H**.

Analysis of Rectangular Waveguide



A rectangular waveguide with perfectly conducting walls, filled with a lossless material.

- For a lossless medium, Maxwell's equations in phasor form become,

$$\nabla^2 \mathbf{E}_s + k^2 \mathbf{E}_s = 0 \qquad \text{…………………(1)}$$

$$\nabla^2 \mathbf{H}_s + k^2 \mathbf{H}_s = 0 \qquad \text{…………………(2)}$$

Where,
$$k = \omega\sqrt{\mu\varepsilon} \qquad \text{…………………(3)}$$

the time factor is $e^{j\omega t}$

$$\mathbf{E}_s = (E_{xs}, E_{ys}, E_{zs}) \qquad \text{and} \qquad \mathbf{H}_s = (H_{xs}, H_{ys}, H_{zs})$$

For the z-component, for example, eq. (1) becomes

$$\frac{\partial^2 E_{zs}}{\partial x^2} + \frac{\partial^2 E_{zs}}{\partial y^2} + \frac{\partial^2 E_{zs}}{\partial z^2} + k^2 E_{zs} = 0 \quad \text{....................(4)}$$

Eqn(4) can be solved by separation of variables (product solution)

Let $$E_{zs}(x, y, z) = X(x)\, Y(y)\, Z(z) \quad \text{....................(5)}$$

where *X(x), Y(y),* and *Z(z)* are functions of x, *y,* and *z*.

Substituting eq. (5) into eq. (4) and dividing by XYZ,

$$\therefore \quad \frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = -k^2 \quad \text{....................(6)}$$

Since the variables are independent, each term in eq. (6) must be constant, so the equation can be written as,

$$\therefore \quad -k_x^2 - k_y^2 + \gamma^2 = -k^2 \quad \text{....................(7)}$$

where $-k_x^2$, $-k_y^2$, and γ^2 are separation constants.
Thus, eq. (6) is separated as,

$$X'' + k_x^2 X = 0 \quad \text{....................(8a)}$$

$$Y'' + k_y^2 Y = 0 \quad \text{....................(8b)}$$

$$Z'' - \gamma^2 Z = 0 \quad \text{....................(8c)}$$

obtain the solution to eq. (8) as,

$$X(x) = c_1 \cos k_x x + c_2 \sin k_x x \quad \text{......................(9a)}$$

$$Y(y) = c_3 \cos k_y y + c_4 \sin k_y y \quad \text{......................(9b)}$$

$$Z(z) = c_5 e^{\gamma z} + c_6 e^{-\gamma z} \quad \text{......................(9c)}$$

Substituting eq. (9) into eq. (5) gives,

$$E_{zs}(x, y, z) = (c_1 \cos k_x x + c_2 \sin k_x x)(c_3 \cos k_y y + c_4 \sin k_y y)(c_5 e^{\gamma z} + c_6 e^{-\gamma z}) \quad \text{...............(10)}$$

As usual, assume that the wave propagates along the waveguide in the +z-direction, the multiplicative constant $c_5 = 0$ because the wave has to be finite at infinity [i.e.,E_{zs}(x, y, z = ∞) = 0].

Hence eq. (10) is reduced to,

$$E_{zs}(x, y, z) = (A_1 \cos k_x x + A_2 \sin k_x x)(A_3 \cos k_y y + A_4 \sin k_y y)e^{-\gamma z}$$

...............(11)

where $A_1 = c_1.c_6$, $A_2 = c_2.c_6$, and so on.
By taking similar steps, get solution of z-component of eq. (2) as,

$$H_{zs}(x, y, z) = (B_1 \cos k_x x + B_2 \sin k_x x)(B_3 \cos k_y y + B_4 \sin k_y y)e^{-\gamma z}$$

...............(12)

Instead of solving for other field component E_{xs} E_{ys} H_{xs} and H_{ys} in eqs. (1) and (2) in the same manner, simply use Maxwell's equations to determine them from E_{zs} and H_{zs} .

From

$$\nabla \times \mathbf{E}_s = -j\omega\mu\mathbf{H}_s$$

$$\nabla \times \mathbf{H}_s = j\omega\varepsilon\mathbf{E}_s$$

Obtain

$$\frac{\partial E_{zs}}{\partial y} - \frac{\partial E_{ys}}{\partial z} = -j\omega\mu H_{xs} \quad \text{..............(13a)}$$

$$\frac{\partial H_{zs}}{\partial y} - \frac{\partial H_{ys}}{\partial z} = j\omega\varepsilon E_{xs} \quad \text{..............(13b)}$$

$$\frac{\partial E_{xs}}{\partial z} - \frac{\partial E_{zs}}{\partial x} = j\omega\mu H_{ys} \quad \text{..............(13c)}$$

$$\frac{\partial H_{xs}}{\partial z} - \frac{\partial H_{zs}}{\partial x} = j\omega\varepsilon E_{ys} \quad \text{..............(13d)}$$

$$\frac{\partial E_{ys}}{\partial x} - \frac{\partial E_{xs}}{\partial y} = -j\omega\mu H_{zs} \quad \text{..............(13e)}$$

$$\frac{\partial H_{ys}}{\partial x} - \frac{\partial H_{xs}}{\partial y} = j\omega\varepsilon E_{zs} \quad \text{..............(13f)}$$

Express E_{xs} E_{ys} H_{xs} and H_{ys} in terms of E_{zs} and H_{zs}.
For E_{xs} for example, combine eqs. (13b) and (13c) and obtain,

$$j\omega\varepsilon E_{xs} = \frac{\partial H_{zs}}{\partial y} + \frac{1}{j\omega\mu}\left(\frac{\partial^2 E_{xs}}{\partial z^2} - \frac{\partial^2 E_{zs}}{\partial x \partial z}\right) \qquad \text{..............(14)}$$

From eqs. (11) and (12), it is clear that all field components vary with z according to $e^{-\gamma z}$, that is,

$$E_{zs} \sim e^{-\gamma z}, \qquad E_{xs} \sim e^{-\gamma z}$$

$$\therefore \quad \frac{\partial E_{zs}}{\partial z} = -\gamma E_{zs}, \qquad \frac{\partial^2 E_{xs}}{\partial z^2} = \gamma^2 E_{xs}$$

$$\therefore \; j\omega\varepsilon E_{xs} = \frac{\partial H_{zs}}{\partial y} + \frac{1}{j\omega\mu}\left(\gamma^2 E_{xs} + \gamma\frac{\partial E_{zs}}{\partial x}\right)$$

$$\therefore \quad -\frac{1}{j\omega\mu}(\gamma^2 + \omega^2\mu\varepsilon)\, E_{xs} = \frac{\gamma}{j\omega\mu}\frac{\partial E_{zs}}{\partial x} + \frac{\partial H_{zs}}{\partial y}$$

Let $h^2 = \gamma^2 + \omega^2\mu\varepsilon = \gamma^2 + k^2$,

$$\therefore \quad E_{xs} = -\frac{\gamma}{h^2}\frac{\partial E_{zs}}{\partial x} - \frac{j\omega\mu}{h^2}\frac{\partial H_{zs}}{\partial y}$$

Similar manipulations of eq. (13) yield expressions for E_{ys} H_{xs} and H_{ys} in terms of E_{zs} and H_{zs}. Thus,

$$\therefore \quad E_{xs} = -\frac{\gamma}{h^2}\frac{\partial E_{zs}}{\partial x} - \frac{j\omega\mu}{h^2}\frac{\partial H_{zs}}{\partial y} \qquad \text{..............(15a)}$$

$$\therefore \quad E_{ys} = -\frac{\gamma}{h^2}\frac{\partial E_{zs}}{\partial y} - \frac{j\omega\mu}{h^2}\frac{\partial H_{zs}}{\partial x} \qquad \text{..............(15b)}$$

$$\therefore \quad H_{xs} = \frac{j\omega\varepsilon}{h^2}\frac{\partial E_{zs}}{\partial y} - \frac{\gamma}{h^2}\frac{\partial H_{zs}}{\partial x} \qquad \text{...............(15c)}$$

$$\therefore \quad H_{ys} = -\frac{j\omega\varepsilon}{h^2}\frac{\partial E_{zs}}{\partial x} - \frac{\gamma}{h^2}\frac{\partial H_{zs}}{\partial y} \qquad \text{...............(15d)}$$

Where, $h^2 = \gamma^2 + k^2 = k_x^2 + k_y^2$ ....(16)

Points to Ponder

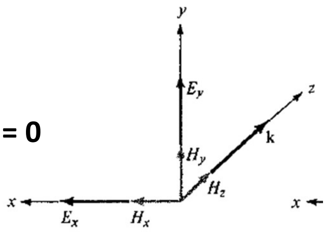
- From eqs. (11), (12), and (15), notice that there are different types of field patterns or configurations.
- Each of distinct field patterns is called a <u>Mode</u>, with 4 different mode categories.
- $E_{zs} = 0 = H_{zs}$ (**TEM** mode): This is the <u>Transverse Electro Magnetic</u> (TEM) mode, in which both the **E** and **H** fields are transverse to the direction of wave propagation.
- From eq. (15), all field components vanish for $E_{zs} = 0 = H_{zs}$.
- So, a rectangular waveguide cannot support TEM mode.

- $E_{zs} \neq 0$, $H_{zs} \neq 0$ (**HE** modes): This is the case when neither **E** nor **H** field is transverse to the direction of wave propagation. They are called <u>hybrid</u> modes.

Points to Ponder

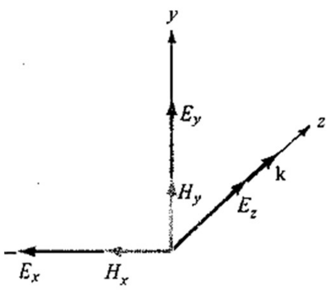
- $E_{zs} = 0$, $H_{zs} \neq 0$ (**TE** modes): Here, the remaining components (E_{xs} and E_{ys}) of the electric field are transverse to the direction of propagation $\mathbf{a}_z$.
- Under this condition, fields are said to be in Transverse Electric (TE) modes.

TE mode E_z = 0



Points to Ponder

- $E_{zs} \neq 0$, $H_{zs} = 0$ (**TM** modes): In this case, the **H** field is transverse to the direction of wave propagation. This is Transverse Magnetic (TM) mode.



TM mode, H_z = 0

TRANSVERSE MAGNETIC (TM) MODES

- For this case, the magnetic field has its components transverse (or normal) to the direction of wave propagation.
- This implies that we set H_z = 0 and determine E_x ;E_y ;E_z ;H_x and H_y using eqs. (11) and (15) and the boundary conditions.
- solve for E_z and later determine other field components from E_z.
- At the walls of the waveguide, the tangential components of the **E** field must be continuous; that is,

$$\therefore \quad E_{zs} = 0 \quad \text{at} \quad y = 0 \quad \text{...............(17a)}$$

$$\therefore \quad E_{zs} = 0 \quad \text{at} \quad y = b \quad \text{...............(17b)}$$

$$\therefore \quad E_{zs} = 0 \quad \text{at} \quad x = 0 \quad \text{...............(17c)}$$

$$\therefore \quad E_{zs} = 0 \quad \text{at} \quad x = a \quad \text{...............(17d)}$$

Equations (17a) and (17c) require that $A_1 = 0 = A_3$ in eq. (11), so eq. (11) becomes,

$$\therefore \quad E_{zs} = E_o \sin k_x x \sin k_y y \, e^{-\gamma z} \quad \text{...............(18)}$$

where $E_0 = A_2A_4$

Also eqs. (17b) and (17d) when applied to eq. (18) require that,

$$\sin k_x a = 0, \qquad \sin k_y b = 0 \quad \text{...............(19)}$$

This implies that,

$$k_x a = m\pi, \qquad m = 1, 2, 3, \ldots \quad \text{...........(20a)}$$

$$k_y b = n\pi, \qquad n = 1, 2, 3, \ldots \quad \text{...........(20b)}$$

$$\therefore \; k_x = \frac{m\pi}{a}, \qquad k_y = \frac{n\pi}{b} \quad \text{...........(21)}$$

Substituting eq. (21) into eq. (18) gives,

$$\therefore \; E_{zs} = E_o \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) e^{-\gamma z} \quad \text{...........(22)}$$

Obtain other field components from eqs. (22) and (15) bearing in mind that H_{zs}= 0. Thus,

$$\therefore \; E_{xs} = -\frac{\gamma}{h^2}\left(\frac{m\pi}{a}\right) E_o \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) e^{-\gamma z} \quad \text{...........(23a)}$$

$$\therefore \; E_{ys} = -\frac{\gamma}{h^2}\left(\frac{n\pi}{b}\right) E_o \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) e^{-\gamma z} \quad \text{...........(23b)}$$

$$\therefore \; H_{xs} = \frac{j\omega\varepsilon}{h^2}\left(\frac{n\pi}{b}\right) E_o \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) e^{-\gamma z} \quad \text{...........(23c)}$$

$$\therefore \; H_{ys} = -\frac{j\omega\varepsilon}{h^2}\left(\frac{m\pi}{a}\right) E_o \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) e^{-\gamma z} \quad \text{...........(23d)}$$

from eqs. (16) and (21).

$$\therefore \; h^2 = k_x^2 + k_y^2 = \left[\frac{m\pi}{a}\right]^2 + \left[\frac{n\pi}{b}\right]^2 \quad \text{...........(24)}$$

Notice from eqs. (22) and (23) that each set of integers *m* and *n* gives a different field pattern or mode, referred to as TM_{mn} mode, in the waveguide.

Integer *m* equals the number of half-cycle variations in the *x*-direction, and integer *n* is the number of half-cycle variations in the y-direction.

Notice from eqs. (22) and (23) that if (m, n) is (0, 0), (0, n), or (m, 0), all field components vanish.

Thus neither *m* nor *n* can be zero.

Consequently, TM_{11} is the lowest-order mode of all the TM_{mn} modes.

By substituting eq. (21) into eq. (16), obtain the propagation constant,

$$\therefore \quad \gamma = \sqrt{\left[\frac{m\pi}{a}\right]^2 + \left[\frac{n\pi}{b}\right]^2 - k^2} \qquad \text{............(25)}$$

Remember, $k = \omega\sqrt{\mu\varepsilon}$ and $\gamma = \alpha + j\beta$.

CASE A (cut-off):

If $k^2 = \omega^2\mu\varepsilon = \left[\frac{m\pi}{a}\right]^2 + \left[\frac{n\pi}{b}\right]^2$

$$\gamma = 0 \qquad \text{or} \qquad \alpha = 0 = \beta$$

The value of w that causes this is called the *cutoff angular frequency* ω_c; i.e.,

$$\omega_c = \frac{1}{\sqrt{\mu\varepsilon}}\sqrt{\left[\frac{m\pi}{a}\right]^2 + \left[\frac{n\pi}{b}\right]^2} \qquad \text{............(26)}$$

CASE B (Evanescent):

If $k^2 = \omega^2\mu\varepsilon < \left[\frac{m\pi}{a}\right]^2 + \left[\frac{n\pi}{b}\right]^2$

$\gamma = \alpha, \qquad \beta = 0$

In this case, there is no wave propagation at all.
These non-propagating or attenuating modes are said to be *Evanescent.*

CASE C (propagation):

If $k^2 = \omega^2\mu\varepsilon > \left[\frac{m\pi}{a}\right]^2 + \left[\frac{n\pi}{b}\right]^2$

$\gamma = j\beta, \qquad \alpha = 0$

that is, from eq. (25) the phase constant β becomes,

$$\therefore \quad \beta = \sqrt{k^2 - \left[\frac{m\pi}{a}\right]^2 - \left[\frac{n\pi}{b}\right]^2} \qquad \text{............(27)}$$

This is the only case when propagation takes place as all field components will have the factor $e^{-\gamma z} = e^{-j\beta z}$.

Thus for each mode, characterized by a set of integers *m* and *n,* there is a corresponding *cut-off frequency* f_c

The **cutoff frequency** is the **operating frequency** below which attenuation occurs and above which propagation takes place.

The waveguide therefore operates as a high-pass filter. The cutoff frequency is obtained from eq. (26) as,

$$\therefore \quad f_c = \frac{\omega_c}{2\pi} = \frac{1}{2\pi\sqrt{\mu\varepsilon}}\sqrt{\left[\frac{m\pi}{a}\right]^2 + \left[\frac{n\pi}{b}\right]^2}$$

$$\therefore \quad f_c = \frac{u'}{2}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2} \qquad \text{…………(28)}$$

Where, $u' = \dfrac{1}{\sqrt{\mu\varepsilon}}$

Phase velocity of uniform plane wave in the lossless dielectric medium ($\sigma = 0$, μ, ε) filling the waveguide.

The *cutoff wave length* λ is given by,

$$\therefore \quad \lambda_c = \frac{u'}{f_c}$$

$$\therefore \quad \lambda_c = \frac{2}{\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}} \qquad \text{…………(29)}$$

Note from eqs. (28) and (29) that TM_{11} has the lowest cutoff frequency (or the longest cutoff wavelength) of all the TM modes.

The phase constant β in eq. (27) can be written in terms of f_c as,

$$\therefore\ \beta = \omega\sqrt{\mu\varepsilon}\sqrt{1 - \left[\frac{f_c}{f}\right]^2}$$

$$\therefore\ \beta = \beta'\sqrt{1 - \left[\frac{f_c}{f}\right]^2} \qquad \text{............(30)}$$

$\beta' = \omega/u' = \omega\sqrt{\mu\varepsilon}$ →phase constant of uniform plane Wave in the dielectric medium.

Note that γ for evanescent mode can be expressed in terms of f_c,

$$\therefore\ \gamma = \alpha = \beta'\sqrt{\left(\frac{f_c}{f}\right)^2 - 1} \qquad \text{............(30a)}$$

The phase velocity u_p and the wavelength λ in the guide are,

$$\therefore\ u_p = \frac{\omega}{\beta}, \qquad \lambda = \frac{2\pi}{\beta} = \frac{u_p}{f} \qquad \text{............(31)}$$

The intrinsic wave impedance of the mode is obtained from eq. (23) as ($\gamma = j\beta$)

$$\therefore \quad \eta_{TM} = \frac{E_x}{H_y} = -\frac{E_y}{H_x}$$

$$= \frac{\beta}{\omega\varepsilon} = \sqrt{\frac{\mu}{\varepsilon}}\sqrt{1 - \left[\frac{f_c}{f}\right]^2}$$

$$\therefore \quad \eta_{TM} = \eta' \sqrt{1 - \left[\frac{f_c}{f}\right]^2} \quad \text{...........(32)}$$

where $\eta' = \sqrt{\frac{\mu}{\varepsilon}}$ = intrinsic impedance of uniform plane wave in the medium.

Note the difference between u', $\beta'$and γ' and u, $\beta$and γ.

The quantities with prime are wave characteristics of the dielectric medium unbounded by the waveguide.

For example, *u'* would be the velocity of the wave if the waveguide were removed and the entire space were filled with the dielectric.

The quantities without prime are the wave characteristics of the medium bounded by the waveguide.

The integers *m* and *n* indicate the number of half-cycle variations in the *x-y* cross section of the guide.
Thus for a fixed time, the field configuration shown results for TM_{21} mode, for example.

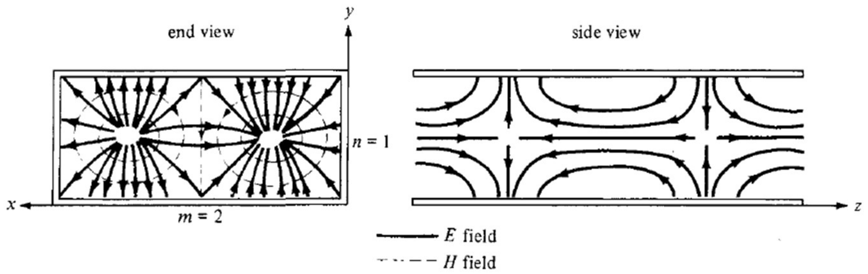


Figure 12.4 Field configuration for TM_{21} mode.

TRANSVERSE ELECTRIC (TE) MODES

- In the TE modes, the electric field is transverse (or normal) to the direction of wave propagation.
- set $E_z = 0$ and determine other field components E_x ; E_y ; H_x ;H_y ; and H_z from eqs. (12) and (15) and the boundary conditions.
- The boundary conditions are obtained from the fact that the tangential components of the electric field must be continuous at the walls of the waveguide; i.e.,

$$\therefore\ E_{xs} = 0 \quad \text{at} \quad y = 0 \qquad \text{............(33a)}$$

$$\therefore\ E_{xs} = 0 \quad \text{at} \quad y = b \qquad \text{............(33b)}$$

$$\therefore\ E_{ys} = 0 \quad \text{at} \quad x = 0 \qquad \text{............(33c)}$$

$$\therefore\ E_{ys} = 0 \quad \text{at} \quad x = a \qquad \text{............(33d)}$$

From eqs. (15) and (33), the boundary conditions can be written as,

$$\therefore\ \frac{\partial H_{zs}}{\partial y} = 0 \quad \text{at} \quad y = 0 \qquad \text{............(34a)}$$

$$\therefore\ \frac{\partial H_{zs}}{\partial v} = 0 \quad \text{at} \quad y = b \qquad \text{............(34b)}$$

$$\therefore\ \frac{\partial H_{zs}}{\partial x} = 0 \quad \text{at} \quad x = 0 \qquad \text{............(34c)}$$

$$\therefore\ \frac{\partial H_{zs}}{\partial x} = 0 \quad \text{at} \quad x = a \qquad \text{............(34d)}$$

Imposing these boundary conditions on eq. (12) yields,

$$H_{zs} = H_o \cos\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) e^{-\gamma z} \qquad \text{............(35)}$$

where $H_o = B_1 B_3$.

Other field components are easily obtained from eqs. (35) and (15) as,

$$E_{xs} = \frac{j\omega\mu}{h^2}\left(\frac{n\pi}{b}\right) H_o \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) e^{-\gamma z} \qquad \text{............(36a)}$$

$$E_{ys} = -\frac{j\omega\mu}{h^2}\left(\frac{m\pi}{a}\right) H_o \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) e^{-\gamma z} \qquad \text{......(36b)}$$

$$H_{xs} = \frac{\gamma}{h^2}\left(\frac{m\pi}{a}\right) H_o \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) e^{-\gamma z} \quad \text{............(36c)}$$

$$H_{ys} = \frac{\gamma}{h^2}\left(\frac{n\pi}{b}\right) H_o \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) e^{-\gamma z} \quad \text{............(36d)}$$

Notice that each set of integers *m* and *n* gives a different field pattern or mode, referred to as TE_{mn} mode, in the waveguide.

Integer *m* equals the number of half-cycle variations in the *x*-direction, and integer *n* is the number of half-cycle variations in the y-direction.

Also notice that (m, n) cannot be (0, 0), all field components vanish but can be (0, 1), or (1, 0).

Thus neither *m* nor *n* can be zero at same time.

So, TE_{01} or TE_{10} is the lowest-order mode of all the TE_{mn} modes depending on the values of a and *b,* the dimensions of the guide.

TE_{10} is called the *dominant mode* of **the** waveguide and is of practical importance.

The cutoff frequency for the TE_{10} mode is obtained from eq. (28) as (m = 1, n = 0),

$$f_{c_{10}} = \frac{u'}{2a} \quad \text{............(37)}$$

and the cutoff wavelength for TE_{10} mode is obtained from eq. (29) as,

$$\lambda_{c_{10}} = 2a \quad \text{............(38)}$$

- The dominant mode is the mode with the lowest cutoff frequency (or longest cutoff wavelength).

Also note that any EM wave with frequency $f < f_{C_{10}}$ (or $\lambda > \lambda c_{10}$) will not be propagated in the guide.

The intrinsic impedance for TE mode is given by,

From eq. (36), it is evident that (γ = **j**β),

$$\eta_{\mathrm{TE}} = \frac{E_x}{H_y} = -\frac{E_y}{H_x} = \frac{\omega\mu}{\beta} \qquad = \sqrt{\frac{\mu}{\varepsilon}}\frac{1}{\sqrt{1-\left[\frac{f_c}{f}\right]^2}}$$

$$\eta_{TE} = \frac{\eta'}{\sqrt{1-\left[\frac{f_c}{f}\right]^2}} \qquad \text{…………(39)}$$

A rectangular waveguide with dimensions $a = 2.5$ cm, $b = 1$ cm is to operate below 15.1 GHz. How many TE and TM modes can the waveguide transmit if the guide is filled with a medium characterized by $\sigma = 0$, $\varepsilon = 4\,\varepsilon_o$, $\mu_r = 1$? Calculate the cutoff frequencies of the modes.

Sol:

The cutoff frequency is given by

$$f_{c_{mn}} = \frac{u'}{2}\sqrt{\frac{m^2}{a^2}+\frac{n^2}{b^2}}$$

where $a = 2.5$b or $a/b = 2.5$, and

$$u' = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{c}{\sqrt{\mu_r\varepsilon_r}} = \frac{c}{2}$$

$$f_{c_{mn}} = \frac{c}{4a}\sqrt{m^2+\frac{a^2}{b^2}n^2} \qquad = \frac{3\times10^8}{4(2.5\times10^{-2})}\sqrt{m^2+6.25n^2}$$

$$f_{c_{mn}} = 3\sqrt{m^2 + 6.25n^2} \text{ GHz}$$

For TE_{01} mode ($m = 0, n = 1$), $f_{c_{01}} = 3(2.5) = 7.5$ GHz

TE_{02} mode ($m = 0, n = 2$), $f_{c_{02}} = 3(5) = 15$ GHz

TE_{03} mode, $f_{c_{03}} = 3(7.5) = 22.5$ GHz

Thus for $f_{c_{mn}} < 15.1$ GHz, the maximum $n = 2$. We now fix n and increase m until $f_{c_{mn}}$ is greater than 15.1 GHz.

For TE_{10} mode ($m = 1, n = 0$), $f_{c_{10}} = 3$ GHz

TE_{20} mode, $f_{c_{20}} = 6$ GHz

TE_{30} mode, $f_{c_{30}} = 9$ GHz

TE_{40} mode, $f_{c_{40}} = 12$ GHz

TE_{50} mode, $f_{c_{50}} = 15$ GHz (the same as for TE_{02})

TE_{60} mode, $f_{c_{60}} = 18$ GHz.

that is, for $f_{c_{mn}} < 15.1$ GHz, the maximum $m = 5$. Now that we know the maximum m and n, we try other possible combinations in between these maximum values.

For TE_{11}, TM_{11} (degenerate modes), $f_{c_{11}} = 3\sqrt{7.25} = 8.078$ GHz

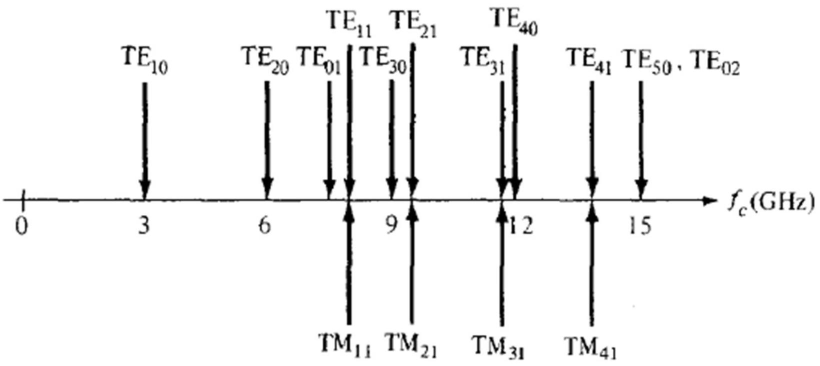
TE_{21}, TM_{21}, $f_{c_{21}} = 3\sqrt{10.25} = 9.6$ GHz

TE_{31}, TM_{31}, $f_{c_{31}} = 3\sqrt{15.25} = 11.72$ GHz

TE_{41}, TM_{41}, $f_{c_{41}} = 3\sqrt{22.25} = 14.14$ GHz

TE_{12}, TM_{12}, $f_{c_{12}} = 3\sqrt{26} = 15.3$ GHz

Those modes whose cutoff frequencies are ≤ 15.1 GHz will be transmitted i.e., 11 TE modes & 4 TM modes.



Cutoff frequencies of rectangular waveguide with *a = 2.5b;*

Consider the waveguide of Example .1. Calculate the phase constant, phase velocity and wave impedance for TE_{10} and TM_{11} modes at the operating frequency of 15 GHz.

In a rectangular waveguide for which $a = 1.5$ cm, $b = 0.8$ cm, $\sigma = 0$, $\mu = \mu_o$, and $\varepsilon = 4\varepsilon_o$,

$$H_x = 2 \sin\left(\frac{\pi x}{a}\right) \cos\left(\frac{3\pi y}{b}\right) \sin(\pi \times 10^{11} t - \beta z) \text{ A/m}$$

Determine

(a) The mode of operation
(b) The cutoff frequency
(c) The phase constant β
(d) The propagation constant γ
(e) The intrinsic wave impedance η.

(a)

$m = 1$, $n = 3$; that is, the guide is operating at TM_{13} or TE_{13}.

choose TM_{13} mode

(b)

$$f_{c_{mn}} = \frac{u'}{2}\sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}$$

$$u' = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{c}{\sqrt{\mu_r\varepsilon_r}} = \frac{c}{2}$$

$$f_{c_{13}} = \frac{c}{4}\sqrt{\frac{1}{[1.5 \times 10^{-2}]^2} + \frac{9}{[0.8 \times 10^{-2}]^2}}$$

$$= \frac{3 \times 10^8}{4}(\sqrt{0.444 + 14.06}) \times 10^2 = 28.57 \text{ GHz}$$

(c) $$\beta = \omega\sqrt{\mu\varepsilon}\sqrt{1 - \left[\frac{f_c}{f}\right]^2} = \frac{\omega\sqrt{\varepsilon_r}}{c}\sqrt{1 - \left[\frac{f_c}{f}\right]^2}$$

$$\omega = 2\pi f = \pi \times 10^{11} \qquad \text{or} \qquad f = \frac{100}{2} = 50 \text{ GHz}$$

$$\beta = \frac{\pi \times 10^{11}(2)}{3 \times 10^8}\sqrt{1 - \left[\frac{28.57}{50}\right]^2} = 1718.81 \text{ rad/m}$$

(d) $$\gamma = j\beta = j1718.81 \text{ /m}$$

(e) $$\eta_{\text{TM}_{13}} = \eta'\sqrt{1 - \left[\frac{f_c}{f}\right]^2} = \frac{377}{\sqrt{\varepsilon_r}}\sqrt{1 - \left[\frac{28.57}{50}\right]^2} = 154.7\ \Omega$$

POWER TRANSMISSION AND ATTENUATION

To determine power flow in the waveguide, we first find the average Poynting vector

$$\mathcal{P}_{\text{ave}} = \frac{1}{2}\text{Re}\,(\mathbf{E}_s \times \mathbf{H}_s^*) \qquad \text{............(40)}$$

In this case, Poynting vector is along z-direction so that,

$$\mathcal{P}_{\text{ave}} = \frac{1}{2}\text{Re}\,(E_{xs}H_{ys}^* - E_{ys}H_{xs}^*)\mathbf{a}_z \qquad \text{............(41)}$$
$$= \frac{|E_{xs}|^2 + |E_{ys}|^2}{2\eta}\,\mathbf{a}_z$$

where $\eta = \eta_{TE}$ for TE modes or $\eta = \eta_{TM}$ for TM modes

The total average power transmitted across the cross section of the waveguide is,

$$P_{ave} = \int \mathcal{P}_{ave} \cdot d\mathbf{S}$$
$$= \int_{x=0}^{a} \int_{y=0}^{b} \frac{|E_{xs}|^2 + |E_{ys}|^2}{2\eta} \, dy \, dx \qquad \text{............(42)}$$

Of practical importance is the attenuation in a lossy waveguide.

In our analysis thus far, we have assumed lossless waveguides ($\sigma = 0$, $\sigma_c = \infty$) for which $\alpha = 0$, $\gamma = j\beta$.

When the dielectric medium is lossy ($\alpha \neq 0$) and the guide walls are not perfectly conducting, ($\sigma_c \neq \infty$), there is a continuous loss of power as a wave propagates along the guide.

$$P_{ave} = P_o e^{-2\alpha z} \qquad \text{............(43)}$$

In order that energy be conserved, the rate of decrease in P_{ave} must equal the time average power loss P_L per unit length, i.e.,

$$P_L = -\frac{dP_{ave}}{dz} = 2\alpha P_{ave}$$

$$\alpha = \frac{P_L}{2P_{ave}} \qquad \text{............(44)}$$

$$\alpha = \alpha_c + \alpha_d \qquad \text{............(44)}$$

where α_c and α_d are attenuation constants due to ohmic or conduction losses ($\sigma_c \neq \infty$) and dielectric losses ($\sigma \neq 0$), respectively.

$$\gamma = \alpha_d + j\beta_d$$
$$= \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - \omega^2\mu\varepsilon_c} \qquad \text{............(45)}$$

$$\varepsilon_c = \varepsilon' - j\varepsilon'' = \varepsilon - j\frac{\sigma}{\omega} \qquad \text{............(46)}$$

Substituting eq. (46) into eq. (45) and squaring both sides of the equation,

$$\gamma^2 = \alpha_d^2 - \beta_d^2 + 2j\alpha_d\beta_d$$
$$= \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - \omega^2\mu\varepsilon + j\omega\mu\sigma$$

Equating real and imaginary parts,

$$\alpha_d^2 - \beta_d^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - \omega^2\mu\varepsilon \qquad \text{............(47a)}$$

$$2\alpha_d\beta_d = \omega\mu\sigma \qquad \text{or} \qquad \alpha_d = \frac{\omega\mu\sigma}{2\beta_d} \qquad \text{............(47b)}$$

Assuming that $\alpha_d^2 \ll \beta_d^2$, $\alpha_d^2 - \beta_d^2 \simeq -\beta_d^2$,

$$\beta_d = \sqrt{\omega^2\mu\varepsilon - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}$$

$$= \omega\sqrt{\mu\varepsilon}\sqrt{1 - \left(\frac{f_c}{f}\right)^2} \qquad \text{............(48)}$$

Substituting eq. (48) into eq. (47b) gives

$$\alpha_d = \frac{\sigma\eta'}{2\sqrt{1 - \left(\frac{f_c}{f}\right)^2}} \qquad \text{............(49)}$$

where $\eta' = \sqrt{\mu/\varepsilon}$

The determination of α_c for TM_{mn} and TE_{mn} modes is time consuming and tedious.
We shall illustrate the procedure by finding α_c for the TE_{10} mode.
For this mode, only E_y H_x and H_z exist.

For the dominant TE_{10} mode, m = 1 and n = 0,

$$\therefore \quad H_{zs} = H_o \cos\left(\frac{\pi x}{a}\right) e^{-j\beta z} \qquad \text{............(50)}$$

$$\because \quad H_z = \text{Re}\,(H_{zs}e^{j\omega t})$$

$$\therefore \quad H_z = H_o \cos\left(\frac{\pi x}{a}\right)\cos(\omega t - \beta z) \qquad \text{............(51a)}$$

$$E_y = \frac{\omega\mu a}{\pi} H_o \sin\left(\frac{\pi x}{a}\right) \sin(\omega t - \beta z) \quad \text{............(51b)}$$

$$H_x = -\frac{\beta a}{\pi} H_o \sin\left(\frac{\pi x}{a}\right) \sin(\omega t - \beta z) \quad \text{............(51c)}$$

$$E_z = E_x = H_y = 0 \quad \text{............(51d)}$$

$$P_{\text{ave}} = \int_{x=0}^{a} \int_{y=0}^{b} \frac{|E_{ys}|^2}{2\eta} \, dx \, dy = \frac{\omega^2 \mu^2 a^2 H_o^2}{2\pi^2 \eta} \int_0^b dy \int_0^a \sin^2 \frac{\pi x}{a} \, dx$$

$$\therefore \; P_{\text{ave}} = \frac{\omega^2 \mu^2 a^3 H_o^2 b}{4\pi^2 \eta} \quad \text{............(52)}$$

The total power loss per unit length in the walls is,

$$\begin{aligned} \therefore \; P_L &= P_L\big|_{y=0} + P_L\big|_{y=b} + P_L\big|_{x=0} + P_L\big|_{x=a} \\ &= 2(P_L\big|_{y=0} + P_L\big|_{x=0}) \end{aligned} \quad \text{............(53)}$$

since the same amount is dissipated in the walls *y* = 0 and *y* = *b* or *x* = 0 and *x* = *a*.
For the wall *y* = 0,

$$\begin{aligned} \therefore \quad P_L\big|_{y=0} &= \frac{1}{2} \text{Re}\left[\eta_c \int (|H_{xs}|^2 + |H_{zs}|^2) \, dx \right]\Bigg|_{y=0} \\ &= \frac{1}{2} R_s \left[\int_0^a \frac{\beta^2 a^2}{\pi^2} H_o^2 \sin^2 \frac{\pi x}{a} \, dx + \int_0^a H_o^2 \cos^2 \frac{\pi x}{a} \, dx \right] \\ &= \frac{R_s a H_o^2}{4} \left(1 + \frac{\beta^2 a^2}{\pi^2}\right) \end{aligned} \quad \text{............(54)}$$

where R_s is the real part of the intrinsic impedance η_c of the conducting wall.

$$\therefore \quad R_s = \frac{1}{\sigma_c \delta} \qquad \text{............(55)}$$

where δ is the skin depth.
R_s is the skin resistance of the wall;

For the wall x = 0,

$$\therefore \quad P_L\,|_{x=0} = \frac{1}{2}\text{Re}\left[\eta_c \int (|H_{zs}|^2)\,dy\right]|_{x=0} = \frac{1}{2}R_s \int_0^b H_o^2\,dy$$

$$= \frac{R_s b H_o^2}{2} \qquad \text{............(56)}$$

$$\therefore \quad P_L = R_s H_o^2 \left[b + \frac{a}{2}\left(1 + \frac{\beta^2 a^2}{\pi^2}\right)\right]$$

$$\therefore \quad \alpha_c = \frac{R_s H_o^2 \left[b + \frac{a}{2}\left(1 + \frac{\beta^2 a^2}{\pi^2}\right)\right] 2\pi^2 \eta}{\omega^2 \mu^2 a^3 H_o^2 b} \qquad \text{............(57)}$$

It is convenient to express α_c in terms of f and f_c.
After some manipulations, we obtain for the TE_{10} mode,

$$\therefore \quad \alpha_c = \frac{2R_s}{b\eta' \sqrt{1 - \left[\frac{f_c}{f}\right]^2}} \left(\frac{1}{2} + \frac{b}{a}\left[\frac{f_c}{f}\right]^2\right) \qquad \text{............(58)}$$

The total attenuation constant α can be easily obtained.

By following the same procedure, the attenuation constant for the TE_{mn} modes ($n \neq 0$) can be obtained as,

$$\therefore \alpha_c|_{TE} = \frac{2R_s}{b\eta'\sqrt{1-\left[\frac{f_c}{f}\right]^2}}\left[\left(1+\frac{b}{a}\right)\left[\frac{f_c}{f}\right]^2 + \frac{\frac{b}{a}\left(\frac{b}{a}m^2+n^2\right)}{\frac{b^2}{a^2}m^2+n^2}\left(1-\left[\frac{f_c}{f}\right]^2\right)\right] \qquad \text{…………(59)}$$

and for the TM_{mn} modes as,

$$\therefore \alpha_c|_{TM} = \frac{2R_s}{b\eta'\sqrt{1-\left[\frac{f_c}{f}\right]^2}}\frac{(b/a)^3 m^2+n^2}{(b/a)^2 m^2+n^2} \qquad \text{…………(60)}$$

The total attenuation constant α can be easily obtained.